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Abstract: The set of subsums of the series $\sum_{n=1}^{\infty} x_n$ is known to be one of three types: a finite union of intervals, homeomorphic to the Cantor set, or of the type known as a Cantorval. Bartoszewicz, Filipczak and Szymonik have described a family of series which contained all known examples of subsum sets which are Cantorvals. We construct another family of series which produces new examples of subsum sets which are Cantorvals.

Keywords: subsum sets, Cantor set, Cantorval, multigeometric series

MCS: 40A05

1 Introduction

1.1 Notation

We consider an infinite sequence $\{\mathbf{x}\} = \{x_2, x, x_3, \dots\}$ of positive terms.

It is well known that rearranging the terms of an absolutely convergent series does not change the sum of the series. We use $\dot{\mathbf{x}} = \sum_{n=1}^{\infty} x_n$ to denote the series arising from $\{\mathbf{x}\}$. Throughout this paper such a series $\dot{\mathbf{x}}$ will be assumed to be a convergent series of positive terms.

Definition 1.1. Let $\dot{\mathbf{x}}$ be a convergent series of positive terms.

(i) A subsum of $\dot{\mathbf{x}}$ is a number $x \in \mathbb{R}$ such that $x = \sum_{n=1}^{\infty} c_n x_n$ where $c_n \in \{0, 1\}$ for all $n \geq 1$.

(ii) The set $E(\dot{\mathbf{x}}) \subset \mathbb{R}$ is the set of all subsums of $\dot{\mathbf{x}}$.

(iii) $X_n = \sum_{i=n+1}^{\infty} x_i$ is the n-th tail of $\dot{\mathbf{x}}$.

We now define the well-known Cantor set $\mathcal{C}$.

Definition 1.2. We recursively define the following subsets of the interval $[0, 1]$:

(i) $C_1 = \left(\frac{1}{3}, \frac{2}{3}\right)$;

(ii) $C_n = \frac{C_{n-1}}{3} \cup \left(\frac{2}{3} + \frac{C_{n-1}}{3}\right)$ for $n \geq 2$.

The ternary Cantor set is $\mathcal{C} = [0, 1] \setminus \left(\bigcup_{n=1}^{\infty} C_n\right)$.

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Remark 1. Each C_n is a union of disjoint 2^{n-1} disjoint open intervals, each of length $\frac{1}{3^n}$.

1.2 Kakeya's results

The following theorem is found in [1], and refers to results from [2–5].

Theorem 1.1. *For any convergent series of positive terms $\dot{\mathbf{x}}$, $E(\dot{\mathbf{x}})$ is exactly one of the following:*

(i) *a finite union of closed and bounded intervals.*
(ii) *homeomorphic to the Cantor set $\mathcal{C}$.*
(iii) *homeomorphic to the set $\mathcal{C}\bigcup(\bigcup_{n=1}^{\infty} C_{2n-1})$.*

Definition 1.3. A Cantorval is a subset of $\mathbb{R}$ that is homeomorphic to $\mathcal{C}\bigcup(\bigcup_{n=1}^{\infty} C_{2n-1})$.

Note that the term Cantorval is used in a more general sense. (See [6], for instance). Our definition above applies to what is often known as an M-Cantorval.

Remark 2. When discussing Cantorvals, the following equality of sets may be useful:

$$\mathcal{C}\bigcup(\bigcup_{n=1}^{\infty} C_{2n-1}) = [0,1] \setminus (\bigcup_{n=1}^{\infty} C_{2n}).$$

The possibilities (i) and (ii) in Theorem 1.1 were first stated by Kakeya in [2] as early as 1914. Kakeya's results are stated below.

Theorem 1.2 (Kakeya's Results)**.**

(i) *$E(\dot{\mathbf{x}})$ is a finite union of closed and bounded intervals if $x_n \le X_n$ for all but finitely many n.*
(ii) *Furthermore, if $\{\mathbf{x}\}$ is a non-increasing sequence and $E(\dot{\mathbf{x}})$ is a finite union of closed and bounded intervals, then $x_n \le X_n$ for all but finitely many n.*
(iii) *$E(\dot{\mathbf{x}})$ is homeomorphic to $\mathcal{C}$ if $x_n > X_n$ for all but finitely many n.*

From Theorem 1.1 and Kakeya's Results, we can deduce the following corollary.

Corollary 1.2.1. *If $E(\dot{\mathbf{x}})$ is a Cantorval, then $x_n \le X_n$ for infinitely many n and $x_n > X_n$ for infinitely many n.*

It should be noted that these conditions do not guarantee that $E(\dot{\mathbf{x}})$ is a Cantorval. For instance, let $\dot{\mathbf{x}}$ be the series such that $x_{2n-1} = \frac{10}{11^n}$ and $x_{2n} = \frac{1}{11^n}$ for $n \ge 1$. It is the case that $x_n > X_n$ for all odd n, and $x_n \le X_n$ for all even n, but yet the set $E(\dot{\mathbf{x}})$ is homeomorphic to the Cantor set $\mathcal{C}$. This follows from a result in a paper by Z. Nitecki. (See Remark 16 in [7].)

For a convergent series of positive terms, conditions which guarantee that its subsum set is a Cantorval are not known. Bartoszewicz, Filipczak and Szymonik in [1] describe families of series which contain all known examples of series for which the set of subsums is a Cantorval. In particular, they consider multigeometric series, and they construct a family of such series whose subsum sets are Cantorvals. In this paper we extend their result by constructing a different family of multigeometric series whose subsum sets are new examples of Cantorvals.

In Section 2 of this paper we generalize a result from [1] by replacing a hypothesis in which the subsum set of a multigeometric series contains a set of consecutive integers by one in which it contains an arithmetic progression. In Section 3 we prove a result which describes a family of series satisfying the latter hypothesis, and whose subsum sets are Cantorvals. Finally in Section 4 we describe a very simple algorithm for generating infinite families of series whose subsum sets are Cantorvals, and we use it to construct two examples.

A referee of the first draft of this paper pointed the authors to the paper by Banakh, Bartoszewicz, Filipczak and Szymonik [8] which gives much more general sufficient conditions for the subsum set of a multigeometric series to be a Cantorval. In Section 4 we will state some of these conditions and apply them to our two examples. Although [8] does give more general conditions than this paper, these conditions do not completely overlap our results. Furthermore our algorithm for producing Cantorvals is new.

2 The main result in [1] and a generalization

Let $k_1, k_2, \dots k_m$ and q be constants with $0 < q < 1$. Then the sequence

$$(k_1, k_2, \dots k_m, k_1 q, k_2 q, \dots k_m q, k_1 q^2, k_2 q^2, \dots k_m q^2, \dots)$$

is called a multigeometric sequence, and is denoted by $(k_1, k_2, \dots k_m; q)$, and its set of subsums by $E(k_1, k_2, \dots, k_m; q)$. (See [1]). Here is the main result by Bartoszewicz, Filipczak and Szymonik in [1].

Theorem 2.1. *Let $k_1 \geq k_2 \geq \cdots k_m$ be positive integers and let $K = \sum_{i=1}^{m} k_i$.*

Suppose that the set $\left\{\sum_{i=1}^{m} c_i k_i : c_i = 0 \text{ or } c_i = 1\right\}$ contains the numbers $n_0, n_0 + 1, n_0 + 2, \dots n_0 + n$ for some positive integers n_0 and n. Then the following are true.

(i) If $q \geq \frac{1}{n+1}$, then $E(k_1, k_2, \dots, k_m; q)$ contains an interval.

(ii) If $q < \frac{k_m}{K + k_m}$, then $E(k_1, k_2, \dots, k_m; q)$ is not a finite union of intervals.

It follows that if $\frac{1}{n+1} \leq q < \frac{k_m}{K + k_m}$, then $E(k_1, k_2, \dots, k_m; q)$ is a Cantorval. The following theorem generalizes this result.

Theorem 2.2. *Let $k_1 \geq k_2 \geq \cdots k_m$ be positive integers, and let $K = \sum_{i=1}^{\infty} k_i$. Suppose that the set $\left\{\sum_{i=1}^{m} c_i k_i : c_i = 0 \text{ or } c_i = 1\right\}$ contains the numbers $a, a + d, a + 2d, \dots, a + nd$ for some positive integers a, d and n. Then the following are true:*

(i) If $q \geq \frac{1}{n+1}$, then $E(k_1, k_2, \dots, k_m; q)$ contains an interval.

(ii) If $q < \frac{k_m}{K + k_m}$, then $E(k_1, k_2, \dots k_m; q)$ is not a finite union of intervals.

(iii) If $\frac{1}{n+1} \leq q < \frac{k_m}{K + k_m}$, then $E(k_1, k_2, \dots k_m; q)$ is a Cantorval.

Our proofs are very similar to the proofs of the result by Bartoszewicz, Filipczak and Szymonik in [1].

Proof of (i). Consider the multigeometric sequence $(d, d, \dots, d; q)$ with d repeated n times. Let x_r be the r-th term, and let $X_r = \sum_{i=r+1}^{\infty} x_i$. We show that $x_r \leq X_r$ for all r.

For any r which is not a multiple of n, $x_r = x_{r+1}$, and hence $x_r \leq X_r$. Suppose that $r = kn$ for some positive integer k. Then $x_{kn} = dq^{k-1}$, and $X_{kn} = \sum_{i=0}^{\infty} ndq^{k+i} = \frac{ndq^k}{1-q}$. Hence $x_{kn} \leq X_{kn}$ if and only if $dq^{k-1} \leq \frac{ndq^k}{1-q}$, if and only if $\frac{1}{n+1} \leq q$, which we have assumed to be true. Therefore $x_r \leq X_r$ for all r. It follows from (i) of Kakeya's results that $E(d, d, \dots, d; q)$ is a finite union of intervals.

Next we show that $\sum_{n=0}^{\infty} aq^n + E(d, d, \dots, d; q)$ is contained in $E(k_1, k_2, \dots, k_m; q)$.

Let $x \in \sum_{n=0}^{\infty} aq^n + E(d, d, \dots, d; q)$. Then $x = (a + aq + aq^2 + \dots) + d(p_0 + p_1 q + p_2 q^2 + \dots)$ for some $p_i \in \{0, 1, 2, \dots, n\}$, that is, $x = (a + p_0 d) + (a + p_1 d)q + (a + p_2 d)q^2 + \cdots$. By hypothesis, each $(a + p_j d)$ has the form $\sum_{i=1}^{m} c_i k_i$ where $c_i = 0$ or $c_i = 1$. Therefore we have $x \in E(k_1, k_2, \dots k_m; q)$.

We have shown that $E(d, d, \dots, d; q)$ is a finite union of intervals. Since $\sum_{n=0}^{\infty} aq^n + E(d, d, \dots, ; q)$ is a translation of $E(d, d, \dots d; q)$, it is a finite union of intervals. Therefore $E(k_1, k_2, \dots k_m; q)$ contains a finite union of intervals, thus proving (i). □

Proof of (ii). Now suppose that $q < \frac{k_m}{K + k_m}$. We will show that the sequence $(k_1, k_2, \dots k_m; q)$ is non-increasing and that $x_{sm} > X_{sm}$ for all positive integers m.

Recall that $k_1 \geq k_2 \geq \cdots k_m$. Hence, to show that the sequence is non-increasing, it is sufficient to show that $k_m \geq k_1 q$. This is true if and only if $q \leq \frac{k_m}{k_1}$. But $q < \frac{k_m}{K + k_m} = \frac{k_m}{(k_1 + k_2 + \cdots k_m) + k_m} < \frac{k_m}{k_1}$. Therefore the sequence is non-increasing.

Observe that $x_{sm} = k_m q^{s-1}$ and that $X_{sm} = k_m q^{s-1}$ and that $X_{sm} = \frac{Kq^s}{1-q}$. So $x_{sm} > X_{sm}$ if and only if $k_m q^{s-1} > \frac{Kq^s}{1-q}$ if and only if $q < \frac{k_m}{K + k_m}$, which we have supposed to be true. From (ii) of Kakeya's results we conclude that $E(k_1, k_2, \dots, k_m; q)$ is not a finite union of intervals, thus proving (ii). □

Proof of (iii). Now suppose that $\frac{1}{n+1} \leq q < \frac{k_m}{K + k_m}$ Then as previously shown, $E(k_1, k_2, \dots, k_m)$ contains an interval but is not a finite union of intervals. By Theorem 1.1, $E(k_1, k_2, \dots, k_m; q)$ is a Cantorval. □

3 A family of Cantorvals

The statement of Theorem 2 describes series whose subsum sets are Cantorvals, but it does not provide simple examples of such series. The next theorem describes such a family of series.

Theorem 3.1. *Let $(a+2nd, a+(2n-2)d, \dots a+2d, a, d; q)$ be a multigeometric sequence with $2nd < a < (2n+2)d$ and $n \geq 4$. If $\frac{1}{2n+2} \leq q < \min\left(\frac{d}{a}, \frac{a-d}{(n+2)a + (n^2+n)d}\right)$, then $E(a+2nd, a+(2n-2)d, \dots, a+2d, a, d; q)$ is a Cantorval.*

The proof of the theorem is contained in the following three lemmas.

Lemma 3.2. *If $\frac{1}{2n+2} \leq q$, then $E(a+2nd, a+(2n-2)d, \dots, a+2d, a, d; q)$ contains a finite union of intervals.*

Proof. Observe that for the series $(a + 2nd, a + (2n-2)d, \dots, a + 2d, a, d; q)$ the set

$$S = \left\{ \sum_{i=1}^{m} c_i k_i : c_i = 0 \text{ or } c_i = 1 \right\}$$

contains the arithmetic progression $(a, a + d, a + 2d, \dots, a + 2nd, a + (2n+1)d)$. It follows from Theorem 2.2 that if $\frac{1}{2n+2} \leq q$, then $E(a + 2nd, a + (2n-2)d, \dots, a + 2d, a, d; q)$ contains a finite union of intervals. □

Next we want to show that $E(a + 2nd, a + (2n-2)d, \dots, a + 2d, a, d; q)$ is not equal to a finite union of intervals. We will do so by using (ii) of Kakeya's results, which implies that if the sequence is non-increasing and $x_n > X_n$ for infinitely many n, $E(a+2nd, a+(2n-2)d, \dots, a+2d, a, d; q)$ is not a finite union of intervals.

But the terms of the sequence $(a + 2nd, a + (2n-2)d, \dots, a + 2d, a_d, d; q)$ may not be non-increasing. Therefore, in order to apply Kakeya's result, we must first rearrange the terms so that they are non-increasing.

Remark 3. It is important to note that a convergent series $\sum_{n=1}^{\infty} x_n$ of positive terms and any rearrangement of it will have the same subsum sets. To see this, observe that if $x = \sum_{n=1}^{\infty} c_n x_n$ is a subsum of $\sum_{n=1}^{\infty} x_n$, then by rearranging the series we will get a subsum of the rearrangement, and since we have absolute convergence, the rearranged sum is also equal to x. By the same reasoning, a subsum of the rearranged series will be a subsum of $\sum_{n=1}^{\infty} x_n$. Hence any conclusions about the subsum set of the rearranged series will be true for the original series as well.

Lemma 3.3. *If we rearrange the terms of the sequence so that dq^{n-1} comes between $(a+2d)q^n$ and aq^n, then the sequence is non-increasing.*

Proof. Consider the first few terms of the sequence:

$$a + 2nd, a + (2n-2)d, \ldots, a + 2d, a, d, (a + 2nd)q, \ldots, (a + 2d)q, aq, \ldots$$

We move d so that it is between $(a+2d)q$ and aq, and in general we move dq^{n-1} between $(a+2d)q^n$ and aq^n. The first few terms of the rearranged sequence become:

$$a + 2nd, a + (2n-2)d, \ldots, a + 2d, a, (a + 2nd)q, (a + (2n-2)d)q, \ldots,$$
$$(a + 2d)q, \mathbf{d}, aq, (a + 2nd)q^2, (a + (2n-2)d)q^2, \ldots, (a + 2d)q^2, \mathbf{dq}, aq^2, \ldots$$

To show that this rearrangement is non-increasing, it is sufficient to show that:

1. $a \geq (a + 2nd)q$;
2. $(a + 2d)q \geq d$;
3. $d \geq aq$.

We first prove 1. By one of the hypotheses of Theorem 3.1 we have that

$$q \leq \frac{a-d}{(n+2)a + (n^2+n)d}.$$

Hence we see that

$$(a + 2nd)q \leq (a + 2nd)\frac{a-d}{(n+2)a + (n^2+n)d}.$$

By another of the hypotheses of Theorem 3.1, $2nd < a$, and hence $a + 2nd < 2a$. Also $a - d < a$.

Therefore

$$(a + 2nd)q \leq 2a\frac{a}{(n+2)a + (n^2+n)d}. \tag{1}$$

By still another hypothesis of Theorem 3.1, $n \geq 4$. It follows that

$$(n+2)a + (n^2+n)d \geq 6a + 20d > 2a.$$

Combining this with (1) gives

$$(a + 2nd)a \leq 2a\frac{a}{2a} = a$$

thus proving 1.

To prove 2 we see that by the hypotheses of Theorem 3.1, $a > 2nd$ and $q \geq \frac{1}{2n+2}$, and so

$$(a + 2d)q > \frac{2nd + 2d}{2n+2} = d.$$

To prove 3 we see that by yet another of the hypotheses of Theorem 3.1, $q < \frac{d}{a}$, which implies that $d > aq$. □

Lemma 3.4. *The set of subsums of the rearrangement described in Lemma 3.3 is not a finite union of intervals.*

Proof. We will show that in the rearranged series there are infinitely many terms which are strictly greater than their tails. First we show that the term a in the rearranged series is strictly greater than its tail.

Let

$$K = (a + 2nd) + (a + (2n - 2)d) + \cdots + (a + 2d) + a + d.$$

The tail of a is $d + \sum_{n=1}^{\infty} Kq^n = d + \frac{Kq}{1-q}$. Hence a is strictly greater than its tail if and only if $a > d + \frac{Kq}{1-q}$, if and only if $q < \frac{a-d}{a-d+K}$. Now oberve that

$$\begin{aligned} K &= (n + 1)a + d + 2d(1 + 2 + \cdots + n) \\ &= (n + 1)a + d + 2d\left(\frac{n(n+1)}{2}\right) \\ &= (n + 1)a + (n^2 + n + 1)d. \end{aligned}$$

Substituting this value for K in the inequality for q, we get that a is strictly greater than its tail if and only if $q < \frac{a-d}{(n+2)a + (n^2+n)d}$. But this is true by one of the hypotheses of Theorem 3.1.

For every positive integer n, the tail of aq^n is given by

$$dq^n + \sum_{i=1}^{\infty} Kq^{n+i} = dq^n + \frac{Kq^{n+1}}{1-q}.$$

We have shown above that $a > d + \frac{Kq}{1-q}$. It follows that $aq^n > dq^n + \frac{Kq^{n+1}}{1-q}$ for every positive integer n. Therefore there are infinitely many terms which are strictly greater than their tails. It follows from (ii) of Kakeya's results that the subsum set of the rearranged series is not a finite union of intervals. □

Remark 4. In Remark 3, we showed that rearranging the terms of a series with positive terms does not change its set of subsums. It follows from Lemma 3.4 that $E(a + 2nd, a + (2n - 2)d, \ldots, a + 2d, a, d; q)$ is not equal to a finite union of intervals.

We can now give the proof of Theorem 3.1.

Proof of Theorem 3.1. By Lemma 3.2, $E(a + 2nd, a + (2n - 2)d, \ldots, a + 2d, a, d; q)$ contains a finite union of intervals. By Remark 4, $E(a + 2nd, a + (2n - 2)d, \ldots, a + 2d, a, d; q)$ is not equal to a finite union of intervals. Hence by Theorem 1.1, $E(a + 2nd, a + (2n - 2)d, \ldots, a + 2d, a, d; q)$ is a Cantorval. □

4 Two examples

We consider the question of how to construct a family of multigeometric series which satisfies the somewhat complicated hypotheses of Theorem 3.1. We need values of d, a and n which satisfy the following conditions:

(1) $2nd < a < (2n + 2)d$;
(2) $n \geq 4$;
(3) $\frac{1}{2n+2} < \min\left\{\frac{d}{a}, \frac{a-d}{(n+2)a + (n^2+n)d}\right\}$.

Proposition 1. If (1) and (2) are satisfied, then (3) is also satisfied.

Proof. Suppose that (1) and (2) are true. From (1) we see that $a < (2n+2)d$ implies that $\frac{1}{2n+2} < \frac{d}{a}$. By a straightforward calculation we see that (3) is true if and only $\frac{n^2+3n+2}{n} < \frac{a}{d}$. Since $n \geq 4$ and $2nd < a$, we see that

$$\frac{n^2+3n+2}{n} = n+3+\frac{2}{n} < n+3+1 = n+4 \leq 2n < \frac{a}{d}.$$

□

Example 4.1. $(17, 15, 13, 11, 9, 1; q)$ is a Cantorval if $\frac{1}{10} < q < \frac{4}{37}$.

Proof. Let $n = 4$. Then we need $8 < \frac{a}{d} < 10$. We could choose $a = 9$ and $d = 1$, which gives us the sequence $(17, 15, 13, 11, 9, 1; q)$. Then

$$\frac{a-d}{(n+2)a+(n^2+n)d} = \frac{9-1}{(4+2)9+(4^2+4)1} = \frac{4}{37}.$$

By Theorem 3.1, $E(17, 15, 13, 11, 9, 1; q)$ is a Cantorval if $\frac{1}{10} < q < \min\left(\frac{1}{9}, \frac{4}{37}\right)$, or $\frac{1}{10} < q < \frac{4}{37}$.

□

Example 4.2. $E(41, 37, 33, 29, 25, 21, 2; q)$ is a Cantorval if $\frac{1}{12} < q < \frac{19}{207}$.

Proof. Suppose that $n = 5$, so that $10 < \frac{a}{d} < 12$. If we choose $d = 2$ and $a = 21$, we then get the sequence $(41, 37, 29, 25, 21, 2; q)$. Using Theorem 3.1 we see that $E(41, 37, 33, 29, 25, 21, 2; q)$ is a Cantorval if $\frac{1}{12} < q < \min\left(\frac{2}{21}, \frac{19}{207}\right)$, or $\frac{1}{12} < q < \frac{19}{207}$. □

It should be noted that the sequence $(17, 15, 13, 11, 9, 1)$ satisfies the hypotheses of the Bartoszewicz, Filipczak and Szymonik result, since the set $\left\{\sum_{i=1}^{m} c_i k_i : c_i = 0 \text{ or } c_i = 1\right\}$ contains the numbers $9, 10, 11, \ldots, 17, 18$, but the result cannot be used to show that $E(17, 15, 13, 11, 9, 1; q)$ is a Cantorval. With their notation, $n = 10$, $k_m = 1$ and $K = 66$, so that the interval $\left[\frac{1}{n+1}, \frac{k_m}{K+k_m}\right] = \left[\frac{1}{11}, \frac{1}{67}\right]$ is empty.

As promised in Section 1, we shall now state the more general results found in [8]. We begin with some definitions.

Definition 4.1. Let $A \subset \mathbb{R}$ be a compact set containing more than one point.

(i) $\operatorname{diam} A = \sup\{|a-b| : a, b \in A\}$ is the diameter of A.
(ii) $\Delta(A) = \sup\{|a-b| : a, b \in A, (a, b) \cap A = \emptyset\}$. Note that $\Delta(A)$ gives the largest gap in A.
(iii) $I(A) = \frac{\Delta(A)}{\Delta(A) + \operatorname{diam}(A)}$.
(iv) $i(A) = \inf\{I(B) : B \subset A, |B| \geq 2\}$.

Let $k_1 \geq k_2 \geq \cdots \geq k_m$ be positive real numbers and let $S = \left\{\sum_{i=1}^{m} c_i k_i : c_i \in \{0, 1\}\right\}$. Also let $q \in (0, 1)$.

Theorem 4.1. *[8]*

1. *$E(k_1, k_2, \ldots, k_m; q)$ is an interval if and only if $q \geq I(S)$.*
2. *$E(k_1, k_2, \ldots k_m; q)$ contains an interval if $q \geq i(S)$.*
3. *$E(k_1, k_2, \ldots, k_m; q)$ is a Cantor set of zero Lebesgue measure if $q < \frac{1}{|S|}$.*

For Example 4.1, the sequence $(17, 15, 13, 11, 9, 1; q)$, we find that

$$S = \{0, 1\} \cup \{9, 10, 11, \ldots, 18\} \cup \{20, 21, 22, \ldots, 46\} \cup \{48, 49, 50, \ldots, 57\} \cup \{65, 66\}.$$

It follows that $\operatorname{diam} S = 66, \Delta(S) = 8, I(S) = \frac{4}{37}$, and $i(S) = \frac{1}{27}$. (Note that the set $B = \{20, 21, 22, \ldots, 46\} \subset S$ and $I(B) = \frac{1}{27}$.) Finally $|S| = 51$. By Theorem 4.1, we see that $E(17, 15, 13, 11, 9, 1; q)$ is a Cantor set of zero measure if $q < \frac{1}{51}$, contains an interval if $\frac{1}{27} \leq q$, and is an interval if $q \geq \frac{4}{37}$. The proof of Lemma 3.4 in Section 3, together with Remark 4 which follows it, implies that the subsum set $E(a + 2nd, a + (2n - 2)d, \ldots, a + 2d, a, d; q)$ is not a finite union of intervals if $q < \min\left(\frac{d}{a}, \frac{a - d}{(n + 2) + (n^2 + n)d}\right)$. Since $a = 9$, $d = 1$ and $n = 4$, it follows that $E(17, 15, 13, 11, 9, 1; q)$ is not a finite union of intervals if $q < \frac{4}{37}$. Therefore $E(17, 15, 13, 11, 9, 1; q)$ is a Cantor set of zero measure if $q < \frac{1}{51}$, is a Cantorval if $\frac{1}{10} \leq q < \frac{4}{37}$, and is an interval if $q \geq \frac{4}{37}$.

For Example 4.2, the sequence $(41, 37, 33, 29, 25, 21, 2; q)$, we have $\operatorname{diam}(S) = 188, \Delta(S) = 19, I(S) = \frac{19}{207}$, $i(S) = \frac{3}{149}$, and $|S| = 84$. Hence by the same reasoning, $E(41, 37, 33, 29, 25, 21, 2; q)$ is a Cantor set of zero measure if $q < \frac{1}{84}$, is a Cantorval if $\frac{3}{149} \leq q < \frac{19}{207}$ and is an interval if $q \geq \frac{19}{207}$.

References

[1] Bartoszewicz A., Filipczak M., Szymonik E., Multigeometric sequences and Cantorvals, Open Math., 2014, 12, 1000–1007.

[2] Kakeya S., On the partial sums of an infinite series, The Science Reports of the Tôhoku University, 1914, 3, 159–164.

[3] Guthrie J.A., Nymann J.E., The topological structure of the set of subsums of an infinite series, Colloq. Math., 1988, 55, 323–327.

[4] Nymann J.E., Sáenz R.A., The topological structure of the set of P - sums of a sequence, Publ. Math. Debrecen, 1997, 50, 305–316.

[5] Nymann J.E., Sáenz R.A., On a paper of Guthrie and Nymann on subsums of infinite series, Colloq. Math., 2000, 83, 1–4.

[6] Mendes P., Oliveira F., On the topological structure of the arithmetic sum of two Cantor sets, Nonlinearity, 1994, 7, 329–343.

[7] Nitecki Z., Subsum sets: intervals, Cantor sets, and Cantorvals, arXiv:1106.3779v2 [math.HO].

[8] Banakh T., Bartoszewicz A., Filipczak M., Szymonik E., Topological and measure properties of some self-similar sets, Topol. Methods Nonlinear Anal., 2015, 46(2), 1013–1028.