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Local characterization of a class of ruled hypersurfaces in $\mathbb{C}^2$

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ABSTRACT

Let $M^3 \subset \mathbb{C}^2$ be a three times differentiable real hypersurface. The Levi form of M transforms under biholomorphism, and when restricted to the complex tangent space, the skew-hermitian part of the second fundamental form transforms under fractional linear transformation. The surfaces for which these forms are constant multiples of each other were identified in previous work, but when the constant had unit modulus there was a global requirement. Here we give a local characterization of hypersurfaces for which the constant has unit modulus.

1. Introduction

Let r be a defining function for a twice differentiable real hypersurface $M^{2n-1} \subset \mathbb{C}^n$ near $p \in M$. In previous work, the author investigated the quotient $\mathcal{Q}_{r,p}/\mathcal{L}_{r,p}$ defined on M where

$$\mathcal{L}_{r,p} = -\det \begin{pmatrix} r & \frac{\partial r}{\partial \bar{z}_k} \\ \frac{\partial r}{\partial z_j} & \frac{\partial^2 r}{\partial z_j \partial \bar{z}_k} \end{pmatrix} \quad \text{and} \quad \mathcal{Q}_{r,p} = -\det \begin{pmatrix} r & \frac{\partial r}{\partial z_k} \\ \frac{\partial r}{\partial z_j} & \frac{\partial^2 r}{\partial z_j \partial z_k} \end{pmatrix}. \tag{1}$$

If M is Levi nondegenerate, i.e., if $\mathcal{L}_{r,p} \neq 0$, the quotient is well-defined. The quotient also is independent of the choice of defining function and has a modulus that is invariant with respect to fractional linear transformations of $\mathbb{C}^n$. The quotient itself is preserved only by affine maps that have real determinant.

In this manuscript we complete a local characterization of hypersurfaces $M^3 \subset \mathbb{C}^2$ for which $\mathcal{Q}_{r,p}/\mathcal{L}_{r,p}$ is constant. We prove the following:

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Theorem 1. *Let $M^3 \subset \mathbb{C}^2$ be a Levi nondegenerate, three times differentiable hypersurface, and suppose there exists $\theta \in [0, 2\pi)$ so that $\mathcal{Q}_{r,p}/\mathcal{L}_{r,p} = e^{i\theta}$ for all $p \in M$. Then M is the image under an affine map $F(z) = Az + B$, where $0 \neq e^{i\theta/2} \det A \in \mathbb{R}$, of a hypersurface parameterized using differentiable functions $c(t)$, $y(t)$ with $c(0) = y(0) = y'(0) = 0$ over an open set $U \subset \mathbb{R}^3$ containing the origin by*

$$(r, s, t) \in U \to \big(t, 0, y(t), 0\big) + r\big(a(t), 1, b(t), 0\big) + s\big(c(t), 0, d(t), 1\big) \in \mathbb{R}^4 \cong \mathbb{C}^2 \tag{2}$$

with $a(t) = -\int_0^t c'(u)y'(u)\,du$, $b(t) = -\int_0^t c'(u)y'(u)^2\,du$, and $d(t) = \int_0^t c'(u)y'(u)\,du$.

The identification $\mathbb{R}^4 \cong \mathbb{C}^2$ that is used in (2) is given by $(x_1, x_2, x_3, x_4) = (z_1, z_2)$ where $z_1 = x_1 + ix_2$, $z_2 = x_3 + ix_4$.

The converse of Theorem 1 is true, too, in the following sense. If M is an affine image of a hypersurface parameterized by (2) for a twice differentiable function $y(t)$ with $y''(t) \neq 0$ and a differentiable function $c(t)$, then $\mathcal{Q}_{r,p}/\mathcal{L}_{r,p}$ is constant with unit modulus on M. The proof of this statement occurs naturally during the proof of Theorem 1.

The characterizations of surfaces for which $\mathcal{Q}_{r,p}/\mathcal{L}_{r,p} = 0$ and $\mathcal{Q}_{r,p}/\mathcal{L}_{r,p} = \varepsilon$ for $|\varepsilon| \neq 0, 1$ were done in [3] and [5], respectively. In the former case, M was proved to be hermitian quadric (see also Jensen [10] and Detraz and Trépreau [7]), and in the latter case, M was proved to be contained in an affine image of

$$M_\varepsilon \stackrel{\text{def}}{=} \{(z_1, z_2) : (z_1 + \bar{z}_1) + |z_2|^2 + \operatorname{Re}(\varepsilon z_2^2) = 0\}.$$

Theorem 1 addresses the remaining case $|\varepsilon| = 1$ and it strengthens the author's previous result in which M also is assumed to be complete. In that case:

Theorem 2. *(See [4].) Let $M^3 \subset \mathbb{C}^2$ be a complete, Levi non-degenerate, three times differentiable hypersurface, and suppose there exists $\theta \in [0, 2\pi)$ so that for all $p \in M$,*

$$\mathcal{Q}_{r,p} = e^{i\theta}\,\mathcal{L}_{r,p}.$$

Then M is the image under an affine map $F(z) = Az + B$, where $0 \neq \det A \in \mathbb{R}$, of a tubed surface over a complete strongly convex curve in the plane spanned by the $\operatorname{Re}(z_1)$ and $\operatorname{Re}(e^{i\theta/2}z_2)$ directions.

It follows that under the condition $\mathcal{Q}_{r,p}/\mathcal{L}_{r,p} = e^{i\theta}$, the requirement of completeness corresponds with the requirement that $c(t) = 0$ in the conclusion of Theorem 1.

The proof of Theorem 1 completes the local characterization of hypersurfaces $M^3 \subset \mathbb{C}^2$ for which $\mathcal{Q}_{r,p}/\mathcal{L}_{r,p}$ is constant started in [3,5]. There remains the question of the existence of additional hypersurfaces for which $|\mathcal{Q}_{r,p}/\mathcal{L}_{r,p}|$ is constant. Barrett identified some of them in [1]. Are there others? There also remains the question of extending the problem to higher dimensions, perhaps through the consideration of eigenvalues for the quotient of matrices in (1).

The underlying motivation for this work is a desire to understand better how analysis in one complex variable is related to analysis in several complex variables. It already is known that both the Cauchy integral (in one complex variable) and the Cauchy–Leray integral (in several complex variables) are invariant under fractional linear transformations [2]. It is hoped that the consideration of hypersurfaces like the ones described above will help to identify other new connections.

For a study of the projective invariants of a real hypersurface in $\mathbb{C}^n$ following the method of moving frames, see Hammond and Robles [8]. The second order tensors presented there, namely $\mathbb{P}$ and $\mathbb{L}$, are related to $\mathcal{Q}$ and $\mathcal{L}$ as follows. In the local coordinate computation for a hypersurface $M^3 \subset \mathbb{C}^2$ ([8, §4.2]), one finds $P_{11} = -4i\,\mathcal{Q}_{r,p}$ and $P_{1\bar{1}} = +4i\,\mathcal{L}_{r,p}$. It is important to note that the quotient of these quantities is invariant just under the subgroup of affine maps that have real determinant; only the modulus of the quotient is projectively invariant. It is possible there are hypersurfaces besides those in Theorem 1 for which the modulus is identically 1.

2. Local geometry for hypersurfaces with unit quotient

In this section we recall the work from [4] that enables us to reduce the proof of Theorem 1 to the case $\mathcal{Q}_{r,p}/\mathcal{L}_{r,p} = 1$ and to conclude that $\mathcal{Q}_{r,p}/\mathcal{L}_{r,p} = 1$ implies that the rank of the second fundamental form is one.

Recall that by a fractional linear transformation in $\mathbb{C}^n$ we mean a map $F = (f_1, \dots, f_n) : \mathbb{C}^n \to \mathbb{C}^n$ where $f_j = g_j/g_{n+1}$,

$$g_j(z) = a_{j,1}z_1 + \cdots + a_{j,n}z_n + a_{j,n+1},$$

and $\det(a_{j,k})_{j,k=1,\dots,n+1} = 1$. The condition $\det(a_{j,k}) = 1$ acts as a normalization and has no effect the transformation itself.

Let $M^{2n-1} \subset \mathbb{C}^n$ be a twice differentiable hypersurface near $p \in M$ and let $r \in C^2(U)$ be a defining function for M in an open set U that contains p. So $M \cap U = \{r = 0\}$ and $\nabla r|_{M\cap U} \neq 0$. If F is biholomorphic near p, then

$$\mathcal{L}_{r,p} = \mathcal{L}_{r\circ F^{-1},F(p)} \cdot |\det F'(p)|^2,$$

and if F is fractional linear, then

$$\mathcal{Q}_{r,p} = \mathcal{Q}_{r\circ F^{-1},F(p)} \cdot (\det F'(p))^2.$$

In particular, if F is fractional linear then the quotient transforms according to

$$\frac{\mathcal{Q}_{r,p}}{\mathcal{L}_{r,p}} = \frac{\mathcal{Q}_{r\circ F^{-1},F(p)}}{\mathcal{L}_{r\circ F^{-1},F(p)}} \frac{\det F'(p)}{\overline{\det F'(p)}}.$$

Here, $\mathcal{Q}_{r\circ F^{-1},F(p)}/\mathcal{L}_{r\circ F^{-1},F(p)}$ is the quotient computed for the hypersurface $F(M)$ that has defining function $r \circ F^{-1}$. Evidently the quantity is preserved only by the fractional linear transformations for which $\det F'$ is real on M. These are the affine maps $F(z) = Az + B$ with $0 \neq \det A \in \mathbb{R}$.

An argument from [4] that is based on the above transformation law enables us to reduce the proof of Theorem 1 to the case that $\mathcal{Q}_{r,p}/\mathcal{L}_{r,p} = 1$. In particular, the affine map $F(z_1, z_2) = (z_1, e^{i\theta/2}z_2)$ transforms a surface M on which $\mathcal{Q}_{r,p}/\mathcal{L}_{r,p} = e^{i\theta}$ to a surface $F(M)$ on which $\mathcal{Q}_{r\circ F^{-1},F(p)}/\mathcal{L}_{r\circ F^{-1},F(p)} = 1$. If Theorem 1 is proved for surfaces for which the quotient is 1, then $F(M)$ is the image under an affine map $G(w) = Aw+B$ where $0 \neq \det A \in \mathbb{R}$ of a parameterized surface described in (2). Subsequently, M is the image under the affine map $F^{-1} \circ G$ of the same parameterized surface and $0 \neq e^{i\theta/2} \det(F^{-1} \circ G)' \in \mathbb{R}$ as claimed.

To show that $\mathcal{Q}_{r,p}/\mathcal{L}_{r,p} = 1$ implies that the rank of the second fundamental form is one, we choose a defining function r that is normalized so that $|\nabla r| \equiv 2$. Then one has a preferred system of vectors represented in complex coordinates by

$$N = (r_{\bar{1}}, r_{\bar{2}}), \quad JN = (ir_{\bar{1}}, ir_{\bar{2}}), \quad X = (-r_2, r_1), \quad JX = (-ir_2, ir_1). \tag{3}$$

The subscripts and barred-subscripts refer to holomorphic and antiholomorphic partial derivatives, respectively. The vectors form an orthonormal basis of tangent vectors in $\mathbb{C}^2$ along M. With these choices one has

$$\mathcal{L}_{r,p} = \frac{1}{2}\left(II(X,X) + II(JX,JX)\right) \tag{4}$$

$$\mathcal{Q}_{r,p} = \frac{1}{2}\left(II(X,X) - II(JX,JX)\right) - \frac{i}{2}\left(II(X,JX) + II(JX,X)\right) \tag{5}$$

where $II(\cdot,\cdot) : TM \times TM \to \mathbb{R}$ is the symmetric bilinear form on TM representing the second fundamental form.

We mention that if $\nabla = \nabla_X : T\mathbb{R}^4 \to T\mathbb{R}^4$ is the standard flat connection on $\mathbb{R}^4$, then the second fundamental form of M can be expressed by $II(X,Y) = \nabla_X N \cdot Y$ where the dot indicates the summation of products of real coordinates in $\mathbb{R}^4$.

Returning to the work in [4], the condition $\mathcal{Q}_{r,p}/\mathcal{L}_{r,p} = 1$ requires that

$$II(X,JX) = II(JX,X) = II(JX,JX) = 0$$

and the Levi nondegeneracy ensures that $II(X,X) \neq 0$. With these restrictions, it follows via the structural equations of a hypersurface that $II(JN,JX) = II(JX,JN) = 0$ and that the second fundamental form has rank one. (See Lemmas 1, 2, and 3 in [4].) In particular, the second fundamental form can be represented by the 3×3 matrix of real functions

$$\begin{pmatrix} \alpha & \beta & 0 \\ \beta & \lambda & 0 \\ 0 & 0 & 0 \end{pmatrix} \tag{6}$$

where $\alpha\lambda - \beta^2 = 0$ and where the rows and columns correspond with the preferred basis of tangent vectors JN, X, JX.

Conversely, if the second fundamental form is represented by a 3×3 matrix as given in (6) where the rows and columns correspond with the preferred basis of tangent vectors, then $\mathcal{Q}_{r,p}/\mathcal{L}_{r,p} = 1$. This follows from (4) and (5).

We mention that the proof of Theorem 2 found in [4] used a theorem of Hartman and Nirenberg that says a complete hypersurface with zero curvature is cylindrical [9]. By using a suitable affine map, the surface was normalized at a point in order for it to appear as a tubed surface over a (totally) real plane as claimed by the theorem. Upon removal of the requirement of completeness, one still is able to conclude that the hypersurface is foliated by real 2-planes. This is key to the proof given in the next section.

3. Proof of Theorem 1

We proceed with the same notation and simplifications from the last section. In particular, we assume $\mathcal{Q}_{r,p}/\mathcal{L}_{r,p} = 1$ and that the second fundamental form has rank one. Moreover, the second fundamental form is represented by the matrix (6) with rows and columns that correspond with the preferred basis of tangent vectors JN, X, JX defined in (3).

As in [4], we choose $p \in M$ and then apply a translation and special unitary transformation in $\mathbb{C}^2$ so that $p = (0,0)$ and the normal vector at p is $N = (0,1) \in \mathbb{C}^2$. Such transformations are affine and have real determinant. We follow this normalization with a map $F(z) = (z_1 - i\beta_0 z_2/\lambda_0, z_2)$ that also has real determinant and which normalizes the second fundamental form so that at $(0,0)$ it takes value

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & \lambda_0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

where $\lambda_0 \neq 0$ by the Levi nondegeneracy. Upon intersection with $\mathbb{R}\times\mathbb{R} \subset \mathbb{C}\times\mathbb{C}$, the hypersurface has trace which is a convex curve that can be parameterized (locally at the origin) by $t \in \mathbb{R} \to \big(t, 0, y(t), 0\big)$ where $y(0) = 0$, $y'(0) = 0$, and $y''(0) \neq 0$.

To show that the normalized hypersurface M can be expressed as in the statement of Theorem 1, we utilize the fact that any twice differentiable real hypersurface $M^3 \subset \mathbb{R}^4$ whose second fundamental form has rank one is foliated by real 2-planes. This is proved, for instance, in Hartman and Nirenberg [9, Lemma 2, p. 906 and Corollary 2, p. 907] who credit a more general result to Chern and Lashof [6, Lemma 2, p. 314]. Since $y''(0) \neq 0$, the 2-planes extend in directions transverse to the plane containing the convex curve in the last paragraph.

In particular, each point of the curve lies in a real 2-plane contained in the hypersurface whose orthogonal projection onto a plane spanned by the $\operatorname{Im} z_1$ and $\operatorname{Im} z_2$ directions has full rank. So there exist additional functions $a(t)$, $b(t)$, $c(t)$, and $d(t)$ so that the hypersurface is parameterized near the origin by

$$(r, s, t) \in U \to \big(t, 0, y(t), 0\big) + r\big(a(t), 1, b(t), 0\big) + s\big(c(t), 0, d(t), 1\big) \in \mathbb{R}^4 \cong \mathbb{C}^2 \tag{7}$$

where $U \subset \mathbb{R}^3$ is an open set containing the origin. In this way the real 2-plane at $\big(t, 0, y(t), 0\big)$ has independent directions $\langle a(t), 1, b(t), 0\rangle$ and $\langle c(t), 0, d(t), 1\rangle$. The normalization of the tangent plane to M at p requires that $a(0) = b(0) = c(0) = d(0) = 0$.

It remains to be seen that the condition $\mathcal{Q}_{r,p}/\mathcal{L}_{r,p} = 1$ corresponds with the restrictions on $a(t)$, $b(t)$, and $d(t)$ that are described in Theorem 1. For this we consider the second fundamental form applied to the preferred basis of tangent vectors.

Let $\psi : U \to \mathbb{R}^4$ denote the embedding given in (7). Then the tangent space at $\psi(r, s, t) \in M$ is spanned by vectors

$$\frac{\partial\psi}{\partial r} = \langle a(t), 1, b(t), 0\rangle \tag{8}$$

$$\frac{\partial\psi}{\partial s} = \langle c(t), 0, d(t), 1\rangle \tag{9}$$

$$\frac{\partial\psi}{\partial t} = \langle \phi_1(r, s, t), 0, \phi_2(r, s, t), 0\rangle$$

where $\phi_1(r, s, t) = 1 + ra'(t) + sc'(t)$ and $\phi_2(r, s, t) = y'(t) + rb'(t) + sd'(t)$. It follows that the normal vector is

$$\widetilde{N} = \langle -\phi_2, a\phi_2 - b\phi_1, \phi_1, c\phi_2 - d\phi_1\rangle. \tag{10}$$

where dependence on the parameters is suppressed for readability. Using (3), the preferred tangent vectors are

$$J\widetilde{N} = \langle -a\phi_2 + b\phi_1, -\phi_2, -c\phi_2 + d\phi_1, \phi_1\rangle \tag{11}$$

$$\widetilde{X} = \langle -\phi_1, c\phi_2 - d\phi_1, -\phi_2, -a\phi_2 + b\phi_1\rangle \tag{12}$$

$$J\widetilde{X} = \langle -c\phi_2 + d\phi_1, -\phi_1, +a\phi_2 - b\phi_1, -\phi_2\rangle. \tag{13}$$

The tilde on the vectors indicates that the vectors do not have unit length. Since we will impose the condition that certain values for the second fundamental form are zero, it will be enough to work with vectors that are not normalized.

In particular, $II(JN, JX) = 0$, $II(X, JX) = 0$, and $II(JX, JX) = 0$ mean the hypersurface is flat in direction JX. So $J\widetilde{X}$ must be a combination of $\frac{\partial\psi}{\partial r}$ and $\frac{\partial\psi}{\partial s}$. Using (8), (9), and (13), we then find $d(t) = -a(t)$ and

$$J\widetilde{X} = -\phi_1 \frac{\partial \psi}{\partial r} - \phi_2 \frac{\partial \psi}{\partial s}. \tag{14}$$

Our subsequent calculations will use the replacement $d(t) = -a(t)$. Using (10) and (14), we next compute

$$\nabla_{J\widetilde{X}} \widetilde{N} = -\phi_1 \langle -b', ab' - ba', a', cb' + aa' \rangle - \phi_2 \langle a', -aa' - bc', c', -ca' + ac' \rangle \tag{15}$$

and note that $II(JN, JX) = 0$, $II(X, JX) = 0$, and $II(JX, JX) = 0$ mean respectively that

(i) $\nabla_{J\widetilde{X}} \widetilde{N} \cdot J\widetilde{N} = 0$
(ii) $\nabla_{J\widetilde{X}} \widetilde{N} \cdot \widetilde{X} = 0$
(iii) $\nabla_{J\widetilde{X}} \widetilde{N} \cdot J\widetilde{X} = 0$.

Condition (iii) already is met as a consequence of the replacement $d(t) = -a(t)$.

Subsequently, it follows from (11), (12), and (15) that (i) and (ii) are equivalent to

$$(b - c)\big(- b'\phi_1^2 + 2a'\phi_1\phi_2 + c'\phi_2^2 \big) = 0$$
$$(1 + a^2 + bc)\big(- b'\phi_1^2 + 2a'\phi_1\phi_2 + c'\phi_2^2 \big) = 0,$$

respectively. Working locally we have $1 + a^2 + bc \neq 0$ since at the origin this expression has value 1. In particular, (i) and (ii) reduce to the single equation

$$-b'\phi_1^2 + 2a'\phi_1\phi_2 + c'\phi_2^2 = 0$$

which is quadratic in parameters r, s. Expanding further this means

$$\kappa_1(t) + 2r\kappa_2(t) + 2s\kappa_3(t) + r^2\kappa_4(t) + 2rs\kappa_5(t) + s^2\kappa_6(t) = 0$$

for all $(r, s, t) \in U$ where

$$\begin{aligned}
\kappa_1(t) &= b'(t) - y'(t)\big(2a'(t) + c'(t)y'(t)\big) \\
\kappa_2(t) &= -y'(t)\big(a'(t)^2 + b'(t)c'(t)\big) \\
\kappa_3(t) &= a'(t)^2 + b'(t)c'(t) \\
\kappa_4(t) &= -b'(t)\big(a'(t)^2 + b'(t)c'(t)\big) \\
\kappa_5(t) &= a'(t)\big(a'(t)^2 + b'(t)c'(t)\big) \\
\kappa_6(t) &= c'(t)\big(a'(t)^2 + b'(t)c'(t)\big).
\end{aligned}$$

Evidently this is equivalent to the pair of equations

$$a'(t)^2 + b'(t)c'(t) = 0$$
$$b'(t) = y'(t)\big(2a'(t) + c'(t)y'(t)\big).$$

Substituting the latter of these into the former gives

$$\big(a'(t) + c'(t)y'(t)\big)^2 = 0$$

so that $a'(t) = -c'(t)y'(t)$ and then $b'(t) = -c'(t)y'(t)^2$. Since $a(0) = b(0) = 0$, we find by integration that $a(t) = -\int_0^t c'(u)y'(u)\,du$ and $b(t) = -\int_0^t c'(u)y'(u)^2\,du$. Finally, $d(t) = -a(t) = \int_0^t c'(u)y'(u)\,du$ and the proof is complete.

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