4-3-2015

The Kerzman–Stein operator for piecewise continuously differentiable regions

Michael Bolt
Calvin University

Andrew Raich
University of Arkansas

J. Math. Anal. Appl. 413 (2014) 242–249

Contents lists available at ScienceDirect

Journal of Mathematical Analysis and Applications

www.elsevier.com/locate/jmaa

Spectrum of the Kerzman–Stein operator for a family of smooth regions in the plane

Michael Bolt [1]

Department of Mathematics and Statistics, 1740 Knollcrest Circle SE, Calvin College, Grand Rapids, MI 49546-4403, USA

ARTICLE INFO

Article history:
Received 10 September 2012
Available online 27 November 2013
Submitted by L. Fialkow

Keywords:
Kerzman–Stein operator
Cauchy operator
Hilbert–Schmidt norm

ABSTRACT

The Kerzman–Stein operator is the skew-hermitian part of the Cauchy operator defined with respect to an unweighted hermitian inner product on the boundary. For bounded regions with smooth boundary, the Kerzman–Stein operator is compact on the Hilbert space of square integrable functions. Here we give an explicit computation of its Hilbert–Schmidt norm for a family of simply connected regions. We also give an explicit computation of the Cauchy operator acting on an orthonormal basis, and we give estimates for the norms of the Kerzman–Stein and Cauchy operators on these regions. The regions are the first regions that display no apparent Möbius symmetry for which there now is explicit spectral information.

1. Introduction

For a smooth region $\Omega \Subset \mathbb{C}$, Kerzman and Stein studied a certain compact operator $\mathcal{A}$ in relation to the Szegő projection. Let $\mathcal{C}$ be the Cauchy transform on Ω, defined for $f \in L^1(\partial\Omega)$ by

$$\mathcal{C}f(z) = \frac{1}{2\pi i}\int_{\partial\Omega}\frac{f(w)\,dw}{w-z} \tag{1}$$

for $z \in \Omega$ and then by using a nontangential limit of the integral for $z \in \partial\Omega$. Kerzman and Stein showed that the Szegő projection and Cauchy transform are related by $\mathcal{S} = \mathcal{C}(\mathcal{I}+\mathcal{A})^{-1}$ where $\mathcal{I}$ is the identity and $\mathcal{A} = \mathcal{C} - \mathcal{C}^*$ [10].

The Kerzman–Stein operator itself is an integral operator. The integration is performed over the boundary against the kernel

$$A(z,w) = \frac{1}{2\pi i}\left(\frac{T(w)}{w-z} - \frac{\overline{T(z)}}{\overline{w}-\overline{z}}\right)$$

for $z, w \in \partial\Omega$. Here $T(w)$ is the unit tangent vector at w expressed as a complex number with unit length. For smooth regions, the apparent singularities cancel each other and for boundaries of finite length the operator is Hilbert–Schmidt.

Since the operator is skew-hermitian there is a natural problem of computing the spectrum. This was first posed by Kerzman in [9]. For smooth regions, the spectrum is a discrete sequence of imaginary eigenvalues that possibly accumulate at zero. The spectrum is trivial in the case of a disc (in fact, $\mathcal{A} = 0$). It is unknown if there are other regions for which the null space of $\mathcal{A}$ has positive dimension.

E-mail address: mdb7@calvin.edu.

[1] Supported by the National Science Foundation under Grant No. DMS-1002453 and by Calvin College through a Calvin Research Fellowship.

0022-247X/$ – see front matter
http://dx.doi.org/10.1016/j.jmaa.2013.11.049

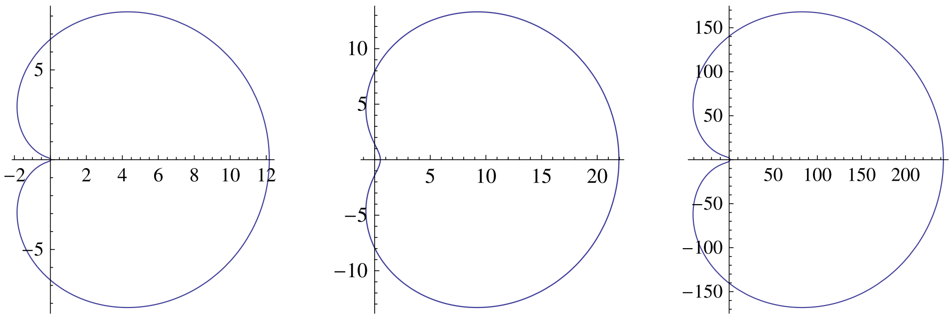

Fig. 1. Regions $\Omega_{\lambda,n}$ for $(\lambda, n) = (1.3, 1), (1.8, 1), (2.0, 2)$. In all cases, $0 \notin \overline{\Omega_{\lambda,n}}$.

Here we give an explicit calculation of the Hilbert–Schmidt norm of the Kerzman–Stein operator for a family of smoothly bounded regions in the plane. The regions are the first regions that display no apparent Möbius symmetry for which there now is explicit spectral information for the Kerzman–Stein operator. Even for the ellipse, the problem only is partially solved [5,7]. We prove:

Theorem 1. *Let $\Omega_{\lambda,n}$ be the image of the unit disc under the map $h(z) = (z + \lambda)^{2n+1}$ where $n \in \mathbb{N}$ and $\lambda > \csc(\pi/(2n+1))$. The Hilbert–Schmidt norm of the Kerzman–Stein operator is given by*

$$\|\mathcal{A}_{\lambda,n}\|_{HS}^2 = \frac{1}{2}\sum_{j,k=1}^{2n} \frac{(-1)^{j+k}\csc\theta_{n,j}\csc\theta_{n,k}}{\sqrt{\lambda^4 - \lambda^2(1+\cot\theta_{n,j}\cot\theta_{n,k}) - \frac{1}{4}(\cot\theta_{n,j} - \cot\theta_{n,k})^2}}$$

where $\theta_{n,j} = \pi j/(2n+1)$.

The hypothesis $\lambda > \csc(\pi/(2n+1))$ guarantees that h is biholomorphic in a neighborhood of the closed unit disc so that the region $\Omega_{\lambda,n}$ is smoothly bounded. (In particular, the translation $z \to z + \lambda$ moves the closed unit disc into the wedge $|\arg z| < \pi/(2n+1)$. Then the map $z \to z^{2n+1}$ acts injectively on the wedge.) The hypothesis also guarantees that the quantity under the square root symbol is positive. Three example regions $\Omega_{\lambda,n}$ are shown in Fig. 1. We point out that in the case $n = 1$, the right-hand side simplifies so that

$$\|\mathcal{A}_{\lambda,1}\|_{HS}^2 = \frac{4}{3}\left(\frac{1}{\sqrt{\lambda^4 - \frac{4}{3}\lambda^2}} - \frac{1}{\sqrt{\lambda^4 - \frac{2}{3}\lambda^2 - \frac{1}{3}}}\right).$$

Recall that the square of the Hilbert–Schmidt norm is the double integral of the squared modulus of the kernel and it coincides with the sum of the sequence of the squared modulus of the eigenvalues.

The methods used here enable one also to compute the Cauchy operator on an orthonormal basis.

Theorem 2. *Let $\Omega_{\lambda,n}$ and h be the same as in Theorem 1, and let $\zeta = \exp\frac{2\pi i}{2n+1}$. Let $\{\phi_k\}_{k\in\mathbb{Z}}$ be the orthonormal basis for $L^2(\partial\Delta)$ given by $\phi_k(z) = z^k/\sqrt{2\pi}$ and let $\Lambda_h : L^2(\partial\Omega_{\lambda,n}) \to L^2(\partial\Delta)$ be the isometry given by $\Lambda_h f = (f \circ h)\sqrt{h'}$. The Cauchy operator for $\Omega_{\lambda,n}$ satisfies*

$$\left(\Lambda_h \circ \mathcal{C}_{\lambda,n} \circ \Lambda_h^{-1}\right)(\phi_{-k})(z) = -\frac{1}{\sqrt{2\pi}}\sum_{j=1}^{2n}\frac{\zeta^{-j(n+k)}}{(z+\lambda(1-\zeta^{-j}))^k}$$

for $k \in \mathbb{N}$. (By the Cauchy integral formula, $(\Lambda_h \circ \mathcal{C}_{\lambda,n} \circ \Lambda_h^{-1})(\phi_k) = \phi_k$ for $k \geqslant 0$.)

Finally, we use Theorems 1 and 2 to provide estimates for the operator norm of the Kerzman–Stein and Cauchy operators on the given regions. These may be helpful for refining the norm estimates by Barrett and the author [1,4].

The Kerzman–Stein operator obeys a transformation law with respect to Möbius transformations, and all other regions for which there is explicit spectral information display some degree of Möbius symmetry that is used in the computations [3]. We mention that there is information regarding the decay of eigenvalues for ellipses that have small eccentricity [5] as well as an observation by Dostanić that the eccentricity of an ellipse can be recovered from the Hilbert–Schmidt norm for ellipses with small eccentricity [7]. For the complicated behavior of ellipses under Möbius transformations, see either Coffman and Frantz [6] or Wilker [11].

It is hoped that the results presented here may be useful for relating the Cauchy and Kerzman–Stein operators to other operators associated to regions in the complex plane. For instance, can these operators be used in special cases to give a direct representation for the Bergman projection? (They always can be used indirectly due to identities that relate the Szegő and Bergman kernels. See Bell [2, p. 94].) It also is hoped that the results may be useful for finding strengthened norm estimates and for finding other regions where the operators can be computed explicitly.

2. Preliminaries

We begin by recalling notation that Bell uses in [2]. Let $\Omega \Subset \mathbb{C}$ be a bounded domain with twice differentiable boundary, and let $T = T(w)$ be the unit tangent vector at $w \in \partial\Omega$ oriented positively with respect to Ω. If ds denotes arc length measure on $\partial\Omega$, then the Kerzman–Stein operator is defined by

$$\mathcal{A}f(z) = \frac{1}{2\pi i}\int_{\partial\Omega}\left(\frac{T(w)}{w-z} - \frac{\overline{T(z)}}{\overline{w}-\bar{z}}\right)f(w)\,ds_w,$$

valid for $f \in L^2(\partial\Omega)$ and $z \in \partial\Omega$. The kernel $A(z,w)$ is bounded at the diagonal—the apparent singularities cancel each other. The space $L^2(\partial\Omega)$ is defined with respect to the hermitian inner product $(f,g)_{\partial\Omega} = \int_{\partial\Omega} f\overline{g}\,ds$.

Since $A(z,w)$ is bounded and $\partial\Omega$ has finite length, it follows that $\mathcal{A}$ is Hilbert–Schmidt and therefore compact on $L^2(\partial\Omega)$. Since $\mathcal{A}$ also is skew-hermitian, the spectral theorem for compact hermitian operators says that the spectrum consists only of imaginary eigenvalues with possible accumulation at zero. If Ω is unbounded or if $\partial\Omega$ has a corner, however, this is not necessarily true. For examples, see [3].

A primary tool that is needed to prove Theorem 1 and Theorem 2 is the following. Let the unit disc be denoted by $\Delta = \{z \in \mathbb{C}\colon\ |z| < 1\}$.

Proposition 1. *Let $h : \Delta \to \Omega$ be a biholomorphism from the unit disc to a smoothly bounded region. Then h' is smooth on $\overline{\Delta}$ and has a well-defined square root. Moreover, the Kerzman–Stein operator for Ω is unitary equivalent to the integral operator defined with respect to arc length measure on $\partial\Delta$ that has kernel*

$$A(z,w) = \frac{1}{2\pi i}\left(\frac{iw\sqrt{h'(w)}\sqrt{h'(z)}}{h(w)-h(z)} - \frac{\overline{iz\sqrt{h'(w)}\sqrt{h'(z)}}}{\overline{h(w)}-\overline{h(z)}}\right) \quad \textit{for } w,z \in \partial\Delta.$$

Proof. A proof for the first claim can be found in Bell [2, p. 42]. For the second claim, we first establish an isometry $L^2(\partial\Omega) \cong L^2(\partial\Delta)$ given by $f \to (f \circ h)\sqrt{h'}$. Taking $f, g \in L^2(\partial\Omega)$, this follows from

$$\int_{\partial\Omega} f\overline{g}\,ds = \int_{\partial\Delta}(f\circ h)\cdot\overline{(g\circ h)}\left|h'\right|ds = \int_{\partial\Delta}(f\circ h)\sqrt{h'}\cdot\overline{(g\circ h)\sqrt{h'}}\,ds.$$

Next, write for $f \in L^2(\partial\Omega)$,

$$\mathcal{A}f(\xi) = \frac{1}{2\pi i}\int_{\partial\Omega}\left(\frac{T(\eta)}{\eta-\xi} - \frac{\overline{T(\xi)}}{\bar{\eta}-\bar{\xi}}\right)f(\eta)\,ds_\eta,$$

and replace $\eta = h(w)$, $T(\eta) = \frac{h'(w)}{|h'(w)|}\cdot iw$, and $ds_\eta = |h'(w)|\,ds_w$, and also $\xi = h(z)$ and $T(\xi) = \frac{h'(z)}{|h'(z)|}\cdot iz$. Then,

$$(\mathcal{A}f)\circ h(z) = \frac{1}{2\pi i}\int_{\partial\Delta}\left(\frac{iw}{h(w)-h(z)}\frac{h'(w)}{|h'(w)|} - \frac{\overline{iz}}{\overline{h(w)}-\overline{h(z)}}\frac{\overline{h'(z)}}{|h'(z)|}\right)(f\circ h)(w)\left|h'(w)\right|ds_w.$$

Multiplying both sides by $\sqrt{h'(z)}$ gives

$$(\mathcal{A}f)\circ h(z)\sqrt{h'(z)} = \frac{1}{2\pi i}\int_{\partial\Delta}\left(\frac{iw\sqrt{h'(w)}\sqrt{h'(z)}}{h(w)-h(z)} - \frac{\overline{iz\sqrt{h'(w)}\sqrt{h'(z)}}}{\overline{h(w)}-\overline{h(z)}}\right)(f\circ h)(w)\sqrt{h'(w)}\,ds_w,$$

and so the proposition is proved. □

3. Proof of Theorem 1

Theorem 1 is proved as follows. Using Proposition 1 we transform the problem from the boundary of the smooth region to the equivalent problem on the unit circle. We expand the kernel using a partial fractions decomposition, and express the square of its modulus as a sum of meromorphic functions that have simple poles away from the unit circle. The integration is accomplished through repeated application of the residue theorem.

So to begin, consider the Cauchy kernel for $\Omega_{\lambda,n}$ as represented on the unit circle in the proof of Proposition 1 by

$$C(z,w)=\frac{1}{2\pi i}\frac{iw\sqrt{h'(w)}\sqrt{h'(z)}}{h(w)-h(z)}=\frac{2n+1}{2\pi}\frac{w(w+\lambda)^n(z+\lambda)^n}{(w+\lambda)^{2n+1}-(z+\lambda)^{2n+1}}.$$

(We suppress subscripts (λ, n) here and in what follows.) The denominator factors as

$$(w+\lambda)^{2n+1}-(z+\lambda)^{2n+1}=\prod_{j=0}^{2n}\big((w+\lambda)-\zeta^j(z+\lambda)\big)$$

where $\zeta=\exp(2\pi i/(2n+1))$. A subsequent partial fractions decomposition gives

$$C(z,w)=\sum_{j=0}^{2n}C_j(z,w)$$

where

$$C_j(z,w)=\frac{1}{2\pi}\frac{w}{\zeta^{nj}}\frac{1}{(w+\lambda)-\zeta^j(z+\lambda)}.$$

The Kerzman–Stein kernel, also represented on the unit circle, is then

$$A(z,w)=C(z,w)-\overline{C(w,z)}=\sum_{j=0}^{2n}C_j(z,w)-\overline{C_j(w,z)}$$

where it is a simple matter to verify that $C_0(z,w)=\overline{C_0(w,z)}$ when $z,w\in\partial\Delta$. It follows that

$$\begin{aligned}\|\mathcal{A}\|_{HS}^2&=\iint\limits_{\partial\Delta\times\partial\Delta}\big|A(z,w)\big|^2\,ds_w\,ds_z=\iint\sum_{j=1}^{2n}\big(C_j(z,w)-\overline{C_j(w,z)}\big)\sum_{k=1}^{2n}\big(\overline{C_k(z,w)}-C_k(w,z)\big)\,ds_w\,ds_z\\&=2\sum_{j,k=1}^{2n}\left(\iint C_j(z,w)\overline{C_k(z,w)}\,ds_w\,ds_z-\operatorname{Re}\iint C_j(z,w)C_k(w,z)\right)ds_w\,ds_z.\end{aligned}\tag{2}$$

We proceed to evaluate the double integrals in the latter expression. Consider first

$$\begin{aligned}\iint C_j(z,w)\,C_k(w,z)\,ds_w\,ds_z&=\left(\frac{1}{2\pi}\right)^2\iint\frac{w}{\zeta^{nj}}\frac{1}{(w+\lambda)-\zeta^j(z+\lambda)}\frac{z}{\zeta^{nk}}\frac{1}{(z+\lambda)-\zeta^k(w+\lambda)}\,ds_w\,ds_z\\&=\left(\frac{1}{2\pi}\right)^2\iint\frac{1}{\zeta^{n(j+k)}}\frac{1}{(w+\lambda)-\zeta^j(z+\lambda)}\frac{\zeta^{-k}}{(w+\lambda)-\zeta^{-k}(z+\lambda)}\,dw\,dz\end{aligned}$$

where in the second step we used $dw=iw\,ds_w$ and $dz=iz\,ds_z$ for $w,z\in\partial\Delta$. Considering the integral in w for fixed z with $|z|=1$, the integrand has singularities only at $w=-\lambda+\zeta^j(z+\lambda), -\lambda+\zeta^{-k}(z+\lambda)$ which both are outside the unit circle. (Indeed, $w+\lambda=\zeta^j(z+\lambda)$ for $|w|,|z|\leqslant 1$ implies $j=0$ using the injectivity of h.) In particular, for $|z|=1$, the integrand is analytic inside the unit disc, and using the Cauchy–Goursat theorem the inner integral is zero.

We are left to evaluate

$$\begin{aligned}\iint C_j(z,w)\overline{C_k(z,w)}\,ds_w\,ds_z&=\left(\frac{1}{2\pi}\right)^2\iint\frac{w}{\zeta^{nj}}\frac{1}{(w+\lambda)-\zeta^j(z+\lambda)}\frac{\overline{w}}{\zeta^{-nk}}\frac{1}{(\overline{w}+\lambda)-\zeta^{-k}(\bar{z}+\lambda)}\,ds_w\,ds_z\\&=\left(\frac{1}{2\pi}\right)^2\frac{\zeta^k}{\zeta^{n(j-k)}}\iint\frac{1}{(w+\lambda)-\zeta^j(z+\lambda)}\frac{1}{w(1+\lambda z)-\zeta^k z(1+\lambda w)}\,dw\,dz.\end{aligned}$$

As before the integrand has a singularity at $w=-\lambda+\zeta^j(z+\lambda)$, but this singularity is outside the unit circle. There also is a singularity when

$$w=\frac{\zeta^k z}{1+\lambda z(1-\zeta^k)}.$$

We claim that when $|z|=1$ this is inside the unit circle. For this, notice that it suffices to verify that $|1+\lambda z(1-\zeta^k)|>1$. Moreover,

$$\big|1+\lambda z\big(1-\zeta^k\big)\big|^2=1+\lambda^2\big|1-\zeta^k\big|^2+2\lambda\operatorname{Re}\big(z\big(1-\zeta^k\big)\big)>1\quad\Leftrightarrow\quad -2\operatorname{Re}\frac{z}{1-\zeta^{-k}}<\lambda.$$

But $|z|=1$ and $1\leqslant k\leqslant 2n$ imply that

$$-2\operatorname{Re}\frac{z}{1-\zeta^{-k}}\leqslant 2\cdot\frac{1}{|1-\zeta^{-k}|}=\csc\left(\frac{\pi k}{2n+1}\right)\leqslant\csc\left(\frac{\pi}{2n+1}\right)<\lambda$$

so the claim is proved.

A first application of the residue theorem can be used then to evaluate the inner integral. This gives

$$\iint C_j(z,w)\,\overline{C_k(z,w)}\,ds_w\,ds_z=\frac{i}{2\pi}\frac{\zeta^k}{\zeta^{n(j-k)}}\int\frac{1}{(w+\lambda)-\zeta^j(z+\lambda)}\cdot\frac{1}{1+\lambda z(1-\zeta^k)}\bigg|_{w=\frac{\zeta^k z}{1+\lambda z(1-\zeta^k)}}dz$$
$$=\frac{i}{2\pi}\frac{\zeta^k}{\zeta^{n(j-k)}}\int\frac{1}{\zeta^k z+[\lambda-\zeta^j(z+\lambda)][1+\lambda z(1-\zeta^k)]}dz.$$

The denominator in the integrand is quadratic in z. So after expanding in powers of z and factoring, we rewrite it as

$$-\lambda\zeta^j\big(1-\zeta^k\big)z^2+\big[\lambda^2\big(1-\zeta^j\big)\big(1-\zeta^k\big)-\big(\zeta^j-\zeta^k\big)\big]z+\lambda\big(1-\zeta^j\big)$$
$$=-\big(1-\zeta^j\big)\big(1-\zeta^k\big)\left[\frac{\lambda\zeta^j}{1-\zeta^j}z^2-\left(\lambda^2-\frac{\zeta^j-\zeta^k}{(1-\zeta^j)(1-\zeta^k)}\right)z-\frac{\lambda}{1-\zeta^k}\right]$$
$$=-\lambda\zeta^j\big(1-\zeta^k\big)\big(z-z_{j,k}^+\big)\big(z-z_{j,k}^-\big)$$

where

$$z_{j,k}^{\pm}=\frac{1-\zeta^j}{2\lambda\zeta^j}\left[\lambda^2-\frac{\zeta^j-\zeta^k}{(1-\zeta^j)(1-\zeta^k)}\pm\sqrt{\lambda^4+\frac{2\lambda^2(\zeta^j+\zeta^k)}{(1-\zeta^j)(1-\zeta^k)}+\left(\frac{\zeta^j-\zeta^k}{(1-\zeta^j)(1-\zeta^k)}\right)^2}\,\right].$$

The quantity under the square root is positive by the identities (3), (4), (5) and inequality (6) that appear below.

It turns out that $z_{j,k}^+$ lies outside the unit circle and $z_{j,k}^-$ lies inside—we return to this shortly. A second application of the residue theorem then gives

$$\iint C_j(z,w)\overline{C_k(z,w)}\,ds_w\,ds_z=\frac{i}{2\pi}\frac{\zeta^k}{\zeta^{n(j-k)}}\int\frac{1}{-\lambda\zeta^j(1-\zeta^k)}\frac{1}{(z-z_{j,k}^+)(z-z_{j,k}^-)}dz$$
$$=\frac{\zeta^k}{\zeta^{n(j-k)}}\frac{1}{\lambda\zeta^j(1-\zeta^k)}\frac{1}{z_{j,k}^--z_{j,k}^+}$$
$$=\frac{\zeta^{(k-j)/2}}{\zeta^{n(j-k)}}\frac{-\zeta^{(j+k)/2}}{(1-\zeta^k)(1-\zeta^j)}\left[\lambda^4+\frac{2\lambda^2(\zeta^j+\zeta^k)}{(1-\zeta^j)(1-\zeta^k)}+\left(\frac{\zeta^j-\zeta^k}{(1-\zeta^j)(1-\zeta^k)}\right)^2\right]^{-\frac{1}{2}}.$$

From $\zeta=\exp(2\pi i/(2n+1))$, one determines that

$$-\frac{\zeta^{(k-j)/2}}{\zeta^{n(j-k)}}=-\zeta^{(k-j)(1/2+n)}=-\exp\big(\pi i(k-j)\big)=(-1)^{j+k+1}.$$

Similarly, one can readily verify the identities

$$\frac{\zeta^j-\zeta^k}{(1-\zeta^j)(1-\zeta^k)}=+\frac{i}{2}(\cot\theta_j-\cot\theta_k),\tag{3}$$
$$\frac{\zeta^j+\zeta^k}{(1-\zeta^j)(1-\zeta^k)}=-\frac{1}{2}(1+\cot\theta_j\cot\theta_k),\tag{4}$$
$$\frac{\zeta^{(j+k)/2}}{(1-\zeta^k)(1-\zeta^j)}=-\frac{1}{4}\csc\theta_j\csc\theta_k\tag{5}$$

where $\theta_j=\pi j/(2n+1)$. (To ease notation we have suppressed the subscript n.) So we obtain

$$\iint C_j(z,w)\overline{C_k(z,w)}\,ds_w\,ds_z=\frac{(-1)^{j+k}\frac{1}{4}\csc\theta_j\csc\theta_k}{\sqrt{\lambda^4-\lambda^2(1+\cot\theta_j\cot\theta_k)-\frac{1}{4}(\cot\theta_j-\cot\theta_k)^2}}.$$

Then Theorem 1 follows from (2) and from the fact that the other double integral (appearing in the last expression of (2)) is zero.

We have left to verify that $z_{j,k}^+$ lies outside the unit circle and that $z_{j,k}^-$ lies inside the unit circle. For this we write

$$z_{j,k}^{\pm} = \frac{\zeta^{-j/2}}{i\lambda}\sin\theta_j\left[\lambda^2 - \frac{i}{2}(\cot\theta_j - \cot\theta_k) \pm \sqrt{\lambda^4 - \lambda^2(1+\cot\theta_j\cot\theta_k) - \frac{1}{4}(\cot\theta_j - \cot\theta_k)^2}\,\right].$$

Using $\lambda > \csc(\pi/(2n+1))$, one can show that

$$\lambda^4 - \lambda^2(1+\cot\theta_j\cot\theta_k) - \frac{1}{4}(\cot\theta_j - \cot\theta_k)^2 > 0, \tag{6}$$

so that after simplification,

$$\left|z_{j,k}^{\pm}\right|^2 = \sin^2\theta_j\left[2\lambda^2 - (1+\cot\theta_j\cot\theta_k) \pm 2\sqrt{\lambda^4 - \lambda^2(1+\cot\theta_j\cot\theta_k) - \frac{1}{4}(\cot\theta_j - \cot\theta_k)^2}\,\right].$$

The fact that $|z_{j,k}^{+}| > 1$ can be seen directly, since ignoring the square root term,

$$\sin^2\theta_j\left[2\lambda^2 - (1+\cot\theta_j\cot\theta_k)\right] > 1 \quad \Leftrightarrow \quad 2\lambda^2 > \csc^2\theta_j + \cos(\theta_j - \theta_k)\csc\theta_j\csc\theta_k.$$

In addition, $\lambda > \csc(\pi/(2n+1))$ ensures both $\lambda^2 > \csc^2\theta_j$ and $\lambda^2 > \csc\theta_j\csc\theta_k$. Meanwhile, the fact that $|z_{j,k}^{-}| < 1$ can be seen indirectly. Since $|z_{j,k}^{+}|^2 > 1$, it suffices to show that

$$\left(1 - \left|z_{j,k}^{+}\right|^2\right)\left(1 - \left|z_{j,k}^{-}\right|^2\right) = 1 - 2\sin^2\theta_j\left[2\lambda^2 - (1+\cot\theta_j\cot\theta_k)\right] + \sin^2\theta_j\csc^2\theta_k < 0.$$

Multiplying through by $\csc^2\theta_j$ one finds that the preceding line is equivalent to

$$\csc^2\theta_j + \csc^2\theta_k + 2\cos(\theta_j - \theta_k)\csc\theta_j\csc\theta_k < 4\lambda^2.$$

But again $\lambda > \csc(\pi/(2n+1))$ ensures all of $\lambda^2 > \csc^2\theta_j$, $\lambda^2 > \csc^2\theta_k$, and $\lambda^2 > \csc\theta_j\csc\theta_k$. So the proof is complete.

4. Proof of Theorem 2

The proof of Theorem 2 uses essentially the same methods as the proof of Theorem 1.

As shown at the beginning of the previous section, the operator $\Lambda_h \circ \mathcal{C} \circ \Lambda_h^{-1}$ has kernel

$$C(z,w) = \sum_{j=0}^{2n} C_j(z,w) = \frac{1}{2\pi}\sum_{j=0}^{2n} \frac{w}{\zeta^{nj}} \frac{1}{(w+\lambda) - \zeta^j(z+\lambda)}$$

where $z \in \Delta$, $w \in \partial\Delta$. The integration is carried out with respect to arc length measure in w.

Using the residue theorem we compute, for $j > 0$,

$$\begin{aligned}
\int_{\partial\Delta} C_j(z,w)\phi_{-k}(w)\,ds_w &= \frac{1}{2\pi}\int_{\partial\Delta} \frac{w}{\zeta^{nj}} \frac{1}{(w+\lambda) - \zeta^j(z+\lambda)} \frac{1}{\sqrt{2\pi}} \frac{1}{w^k}\,ds_w \\
&= \frac{1}{2\pi i}\int_{\partial\Delta} \frac{1}{\zeta^{nj}} \frac{1}{(w+\lambda) - \zeta^j(z+\lambda)} \frac{1}{\sqrt{2\pi}} \frac{1}{w^k}\,dw \\
&= \frac{1}{\sqrt{2\pi}} \frac{1}{\zeta^{nj}} \operatorname{Res}\left(\frac{1}{(w+\lambda) - \zeta^j(z+\lambda)} \frac{1}{w^k}; 0\right) \\
&= \frac{1}{\sqrt{2\pi}} \frac{1}{\zeta^{nj}} \frac{(-1)^{k+1}}{((0+\lambda) - \zeta^j(z+\lambda))^k} \\
&= \frac{1}{\sqrt{2\pi}} \frac{-\zeta^{-j(n+k)}}{(z + \lambda(1-\zeta^{-j}))^k}.
\end{aligned}$$

(The fact that the singularity $w = -\lambda + \zeta^j(z+\lambda)$ lies outside the unit circle was justified in the sentences that followed (2).)

Meanwhile, for $j = 0$,

$$\int_{\partial\Delta} C_0(z,w)\phi_{-k}(w)\,ds_w = \frac{1}{2\pi i}\int_{\partial\Delta} \frac{1}{w-z} \frac{1}{\sqrt{2\pi}} \frac{1}{w^k}\,dw = 0$$

where the last identity holds because the integrand is analytic on the complement of the unit disc. So we conclude that

$$\left(\Lambda_h \circ \mathcal{C} \circ \Lambda_h^{-1}\right)(\phi_{-k})(z) = -\frac{1}{\sqrt{2\pi}} \sum_{j=1}^{2n} \frac{\zeta^{-j(n+k)}}{(z + \lambda(1-\zeta^{-j}))^k}$$

as claimed.

5. Discussion of estimates

In this section we present consequences of Theorem 1 and Theorem 2 for the problem of estimating the norm of the Kerzman–Stein and Cauchy operators acting on $L^2(\partial\Omega)$. We restrict to the case $\Omega = \Omega_{\lambda,1}$.

As already suggested, it follows from the Cauchy integral formula and the boundedness of the Hilbert transform that $\mathcal{C}$ projects $L^2(\partial\Omega)$ to the closed subspace $H^2(\partial\Omega)$ of boundary values of functions holomorphic on Ω. It follows that the norms of $\mathcal{A}$ and $\mathcal{C}$ acting as operators on $L^2(\partial\Omega)$ are related by the identity

$$\|\mathcal{A}\| = \left(\|\mathcal{C}\|^2 - 1\right)^{1/2}. \tag{7}$$

In fact, this identity holds for general projection operators acting on a Hilbert space. A proof can be found in Gerisch [8]. Notice that our definition gives $\mathcal{A} = 2i \operatorname{Im} \mathcal{C}$.

For lower estimates we consider the Cauchy operator applied to one of the basis functions for $L^2(\partial\Omega_{\lambda,1})$. In particular, from Theorem 2 we find that

$$\left(\Lambda_h \circ \mathcal{C}_{\lambda,1} \circ \Lambda_h^{-1}\right)(\phi_{-1})(z) = -\frac{1}{\sqrt{2\pi}}\left(\frac{\zeta^{-2}}{z+\lambda(1-\zeta^{-1})} + \frac{\zeta^{-4}}{z+\lambda(1-\zeta^{-2})}\right),$$

where $\zeta = \exp(2\pi i/3)$, and we compute

$$\begin{aligned}
\left\|\left(\Lambda_h \circ \mathcal{C}_{\lambda,1} \circ \Lambda_h^{-1}\right)(\phi_{-1})\right\|^2 &= \frac{1}{2\pi}\int_{\partial\Delta}\left|\frac{\zeta^{-2}}{z+\lambda(1-\zeta^{-1})} + \frac{\zeta^{-4}}{z+\lambda(1-\zeta^{-2})}\right|^2 ds_z \\
&= \frac{1}{2\pi i}\int_{\partial\Delta}\left(\frac{\zeta^{-2}}{z+\lambda(1-\zeta^{-1})} + \frac{\zeta^{-4}}{z+\lambda(1-\zeta^{-2})}\right) \\
&\quad \cdot \left(\frac{\zeta^{2}}{1+\lambda z(1-\zeta)} + \frac{\zeta^{4}}{1+\lambda z(1-\zeta^{2})}\right) dz \\
&= \frac{\zeta^2}{\lambda(1-\zeta)}\left(\frac{\zeta^{-2}}{z+\lambda(1-\zeta^{-1})} + \frac{\zeta^{-4}}{z+\lambda(1-\zeta^{-2})}\right)\bigg|_{z=-\frac{1}{\lambda(1-\zeta)}} \\
&\quad + \frac{\zeta^4}{\lambda(1-\zeta^2)}\left(\frac{\zeta^{-2}}{z+\lambda(1-\zeta^{-1})} + \frac{\zeta^{-4}}{z+\lambda(1-\zeta^{-2})}\right)\bigg|_{z=-\frac{1}{\lambda(1-\zeta^2)}} \\
&= \frac{1+3\lambda^2}{-1+6\lambda^2-18\lambda^4+27\lambda^6}.
\end{aligned}$$

(The third identity again uses the residue theorem and the fact that $|\lambda(1-\zeta^k)| > 2$; the last identity uses a fair amount of algebraic simplification.) From this it follows that

$$\|\mathcal{C}_{\lambda,1}\| \geqslant \sqrt{1 + \left(\frac{1+3\lambda^2}{-1+6\lambda^2-18\lambda^4+27\lambda^6}\right)}. \tag{8}$$

Indeed, the last estimate follows from the general observation that a Hilbert space projection $P : H \to E$ satisfying $Pf = kg$ for $f \in E^\perp$ and $\|f\| = \|g\| = 1$, also must satisfy $\|P\| \geqslant \sqrt{1+k^2}$. In particular, $P(\frac{1}{\sqrt{1+k^2}}g + \frac{k}{\sqrt{1+k^2}}f) = \sqrt{1+k^2}\,g$.

For upper estimates we recognize that $\mathcal{A}_{\lambda,1}$ is Hilbert–Schmidt and therefore compact. Since also it is skew-hermitian, its spectrum is a discrete except for the possible accumulation at 0. In [5] it was shown that the Kerzman–Stein operator commutes with the anti-linear involution $f \to \overline{fT}$, so the (imaginary) spectrum is symmetric with respect to zero. That is, the eigenvalues appear in pairs $\pm i\alpha_j$, and since $\|\mathcal{A}_{\lambda,1}\|_{HS}^2 = 2\sum_{j\in\mathbb{N}} |i\alpha_j|^2$, it follows that

$$\|\mathcal{A}_{\lambda,1}\| = \max_j |i\alpha_j| \leqslant \sqrt{\frac{2}{3}}\left(\frac{1}{\sqrt{\lambda^4 - \frac{4}{3}\lambda^2}} - \frac{1}{\sqrt{\lambda^4 - \frac{2}{3}\lambda^2 - \frac{1}{3}}}\right)^{1/2}. \tag{9}$$

Finally, using (7) in combination with (8) gives

$$\|\mathcal{A}_{\lambda,1}\| \geqslant \sqrt{\frac{1+3\lambda^2}{-1+6\lambda^2-18\lambda^4+27\lambda^6}}; \tag{10}$$

using (7) in combination with (9) gives

$$\|\mathcal{C}_{\lambda,1}\| \leqslant \left[1 + \frac{2}{3}\left(\frac{1}{\sqrt{\lambda^4 - \frac{4}{3}\lambda^2}} - \frac{1}{\sqrt{\lambda^4 - \frac{2}{3}\lambda^2 - \frac{1}{3}}}\right)\right]^{1/2}. \tag{11}$$

References

[1] David E. Barrett, Michael Bolt, Cauchy integrals and Möbius geometry of curves, Asian J. Math. 11 (1) (2007) 47–53.
[2] Steven R. Bell, The Cauchy Transform, Potential Theory, and Conformal Mapping, Stud. Adv. Math., CRC Press, Boca Raton, FL, 1992.
[3] Michael Bolt, Spectrum of the Kerzman–Stein operator for model domains, Integral Equations Operator Theory 50 (3) (2004) 305–315.
[4] Michael Bolt, A lower estimate for the norm of the Kerzman–Stein operator, J. Integral Equations Appl. 19 (4) (2007) 453–463.
[5] Michael Bolt, Spectrum of the Kerzman–Stein operator for the ellipse, Integral Equations Operator Theory 57 (2) (2007) 167–184.
[6] Adam Coffman, Marc Frantz, Möbius transformations and ellipses, Pi Mu Epsilon J. 12 (6) (2007) 339–345.
[7] Milutin R. Dostanić, The Kerzman–Stein operator for the ellipse, Publ. Inst. Math. (Beograd) (N.S.) 87 (101) (2010) 1–7.
[8] Wolfgang Gerisch, Idempotents, their Hermitian components, and subspaces in position p of a Hilbert space, Math. Nachr. 115 (1984) 283–303.
[9] N. Kerzman, Singular integrals in complex analysis, in: Harmonic Analysis in Euclidean Spaces, Williams Coll., Williamstown, Mass., 1978, in: Proc. Sympos. Pure Math., vol. XXXV, Part 2, Amer. Math. Soc., Providence, RI, 1979, pp. 3–41.
[10] N. Kerzman, E.M. Stein, The Cauchy kernel, the Szegö kernel, and the Riemann mapping function, Math. Ann. 236 (1) (1978) 85–93.
[11] J.B. Wilker, When is the inverse of an ellipse convex?, Util. Math. 17 (1980) 45–50.