

## A GLOBAL CHARACTERIZATION OF TUBED SURFACES IN $\mathbb{C}^2$

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ABSTRACT. Let  $M^3 \subset \mathbb{C}^2$  be a three times differentiable real hypersurface. The Levi form of  $M$  transforms under biholomorphism, and when restricted to the complex tangent space, the skew-Hermitian part of the second fundamental form transforms under Möbius transformations. The surfaces for which these forms are constant multiples of each other were identified in previous work, provided the constant is not unimodular. Here it is proved that if the surface is assumed to be complete and if the constant is unimodular, then the surface is tubed over a strongly convex curve. The converse statement is true, too, and is easily proved.

### 1. INTRODUCTION

Let  $r$  be a defining function for a twice differentiable real hypersurface  $M^{2n-1} \subset \mathbb{C}^n$  near  $p \in M$ . It is a familiar fact that the Levi determinant

$$\mathcal{L}_{r,p} = -\det \begin{pmatrix} r & \frac{\partial r}{\partial \bar{z}_k} \\ \frac{\partial r}{\partial z_j} & \frac{\partial^2 r}{\partial z_j \partial \bar{z}_k} \end{pmatrix}$$

obeys a transformation law under biholomorphism. In [4], the author showed that the related quantity

$$\mathcal{Q}_{r,p} = -\det \begin{pmatrix} r & \frac{\partial r}{\partial z_k} \\ \frac{\partial r}{\partial \bar{z}_j} & \frac{\partial^2 r}{\partial \bar{z}_j \partial z_k} \end{pmatrix}$$

obeys a transformation law, too, provided the biholomorphism is a Möbius transformation. (Möbius transformations are the automorphisms of  $\mathbb{C}\mathbb{P}^n$ .) For a Levi non-degenerate surface, the quotient  $\mathcal{Q}_{r,p}/\mathcal{L}_{r,p}$  then behaves like a Möbius-invariant curvature function, and this motivated the problem of characterizing those surfaces for which this function is constant.

In this paper, we finish the characterization for surfaces in  $\mathbb{C}^2$  and prove the following.

**Theorem 1.** *Let  $M^3 \subset \mathbb{C}^2$  be a complete, Levi non-degenerate, three times differentiable hypersurface, and suppose there exists  $\theta \in [0, 2\pi)$  so that for all  $p \in M$ ,*

$$(1) \quad \mathcal{Q}_{r,p} = e^{i\theta} \mathcal{L}_{r,p}.$$

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Then  $M$  is the image under an affine map of the form  $F(z) = Az + b$ , where  $0 \neq \det A \in \mathbb{R}$ , of a tubed surface over a complete strongly convex curve in the plane spanned by the  $\operatorname{Re}(z_1)$  and  $\operatorname{Re}(e^{i\theta/2}z_2)$  directions.

The converse of Theorem 1 is true, too, and is easily proved. By a tubed surface is meant a surface that has freedom in the  $\operatorname{Im}(z_1)$  and  $\operatorname{Im}(e^{i\theta/2}z_2)$  directions. By a complete planar curve is meant a curve that either is closed or when parameterized by  $(x(t), y(t))$  for  $-\infty < t < +\infty$  with  $x'(t)^2 + y'(t)^2 \neq 0$ , then

$$\lim_{t \rightarrow -\infty} (x^2 + y^2) = \lim_{t \rightarrow +\infty} (x^2 + y^2) = \infty.$$

It is important to note that condition (1) does not depend on the choice of the defining function. That condition, however, is not fully Möbius-invariant. It is preserved only by the subgroup of affine maps that have real determinant as expressed in the statement of the theorem.

The condition  $\mathcal{Q}_{r,p} \equiv 0$ , however, is Möbius-invariant and corresponds with  $M$  being a Hermitian quadric. In fact, this result holds locally. That is, if  $\mathcal{Q}_{r,p} \equiv 0$  but  $M$  is not complete, then  $M$  is contained in a Hermitian quadric [3, 5]. The condition  $\mathcal{Q}_{r,p} \equiv \varepsilon \mathcal{L}_{r,p}$  when  $\varepsilon \in \mathbb{C}$  with  $|\varepsilon| \neq 0, 1$  is also locally well understood. It corresponds with  $M$  being contained in the image of the surface

$$M_\varepsilon \stackrel{\text{def}}{=} \{(z_1, z_2) : (z_1 + \bar{z}_1) + |z_2|^2 + \operatorname{Re}(\varepsilon z_2^2) = 0\}$$

under an affine map with real (and non-zero) determinant [4].

It is not clear if Theorem 1 extends in the obvious way in case  $M$  is not complete. The proof uses an application of the theorem by Hartman and Nirenberg that says that a complete surface with zero curvature is cylindrical. Locally, however, there exist surfaces with zero curvature that are not cylindrical. It seems quite possible that at least formally there might exist analytic hypersurfaces that are non-cylindrical (and non-complete) that satisfy condition (1).

We mention that this work is part of a larger project to understand the Möbius geometry of surfaces and certain Möbius-invariant operators that arise in several complex variables. See, for instance, Barrett [1] and Barrett and Lanzani [2].

## 2. DIFFERENTIAL GEOMETRY OF HYPERSURFACES IN EUCLIDEAN SPACE

The proof of Theorem 1 uses classical differential geometry. We use the following notation, much of which can be found in Hicks [7].

Coordinates  $(z_1, \dots, z_n) \in \mathbb{C}^n$  correspond with coordinates  $(x_1, y_1, \dots, x_n, y_n) \in \mathbb{R}^{2n}$  according to  $z_j = x_j + iy_j$ . Under this identification, the real Euclidean space inherits a complex structure  $J : T\mathbb{R}^{2n} \rightarrow T\mathbb{R}^{2n}$  that corresponds with multiplication by  $i = \sqrt{-1}$  and is given by  $J(\partial_{x_j}) = \partial_{y_j}$ ,  $J(\partial_{y_j}) = -\partial_{x_j}$ . This structure preserves the Euclidean inner product  $\langle \cdot, \cdot \rangle$  on  $T\mathbb{R}^{2n}$ . In fact,  $J^* = -J$  and  $J^2 = -I$ . For  $X \in T\mathbb{R}^{2n}$ , we let  $\bar{d} = \bar{d}_X$  denote the standard flat connection on  $\mathbb{R}^{2n}$ . The complex structure and the connection commute with one another.

The real tangent space of  $M = M^{2n-1}$  is denoted by  $TM$ . The complex tangent space is its codimension-one subspace  $HM = TM \cap J(TM)$ . If  $M$  has defining function  $r$ , then a vector  $X \in H_pM$  can be represented in coordinates by  $s = (s_1, \dots, s_n) \in \mathbb{C}^n$ , where  $\sum r_j(p)s_j = 0$ . The subscripts on  $r$  refer to holomorphic partial derivatives.

Let  $N$  be a unit normal vector on  $M$ . Then the direction orthogonal to  $HM$  in  $TM$  is  $JN$ . For  $X \in TM$ , let  $d = d_X$  be the Riemannian connection that  $M$

inherits as a submanifold of  $\mathbb{R}^n$ . (It is exactly the restriction of  $\bar{d} = \bar{d}_X$  to  $M$ .) Like  $\bar{d}$ , the connection is symmetric and metric, so  $[X, Y] = d_X Y - d_Y X$  for  $X, Y \in TM$ , and  $X\langle Y, Z \rangle = \langle d_X Y, Z \rangle + \langle Y, d_X Z \rangle$  for  $X, Y, Z \in TM$ .

The Weingarten map is the operator  $S : TM \rightarrow TM$  given by  $S(X) = \bar{d}_X N$ . This operator is self-adjoint. Related to  $S$  is the second fundamental form. This is the symmetric bilinear form  $b(X, Y) = \langle S(X), Y \rangle = \langle \bar{d}_X N, Y \rangle$ . The main structural equation for a hypersurface in Euclidean space is the Codazzi equation. It says that if  $X, Y \in TM$ , then  $d_X S(Y) - d_Y S(X) - S([X, Y]) = 0$ . This vector equation describes the compatibility conditions between the induced metric and the second fundamental form for a hypersurface in Euclidean space.

### 3. THE GEOMETRIC SIGNIFICANCE OF $\mathcal{L}_{r,p}$ AND $\mathcal{Q}_{r,p}$

We briefly recall some facts from [4] that are also relevant to the statement and proof of Theorem 1.

By a Möbius transformation in  $\mathbb{C}^n$  we mean a fractional linear transformation  $F = (f_1, \dots, f_n) : \mathbb{C}^n \rightarrow \mathbb{C}^n$ , where  $f_j = g_j/g_{n+1}$ ,

$$g_j(z) = a_{j,1}z_1 + \dots + a_{j,n}z_n + a_{j,n+1},$$

and  $\det(a_{j,k})_{j,k=1,\dots,n+1} = 1$ . The condition  $\det(a_{j,k}) = 1$  acts as a normalization and has no effect on the transformation itself.

Let  $M^{2n-1} \subset \mathbb{C}^n$  be a twice differentiable hypersurface near  $p \in M$  and let  $r \in C^2(U)$  be a defining function for  $M$  in an open set  $U$  that contains  $p$ . So  $M \cap U = \{r = 0\}$  and  $\nabla r|_{M \cap U} \neq 0$ . If  $w = F(z)$  is biholomorphic on  $U$ , then  $M' = F(M \cap U)$  is twice differentiable, it has defining function  $r \circ F^{-1}$  near  $F(p)$ , and

$$\mathcal{L}_{r,p} = \mathcal{L}_{r \circ F^{-1}, F(p)} \cdot |\det F'(p)|^2.$$

Furthermore, if  $F$  is a Möbius transformation, then

$$\mathcal{Q}_{r,p} = \mathcal{Q}_{r \circ F^{-1}, F(p)} \cdot (\det F'(p))^2.$$

It follows that if  $M$  is Levi non-degenerate and  $F$  is a Möbius transformation, then

$$\frac{\mathcal{Q}_{r,p}}{\mathcal{L}_{r,p}} = \frac{\mathcal{Q}_{r \circ F^{-1}, F(p)} \det F'(p)}{\mathcal{L}_{r \circ F^{-1}, F(p)} \det F'(p)}.$$

The affine transformations that are described in Theorem 1 are exactly the transformations for which  $\det F'$  is non-zero and real. It follows that they are also the ones through which  $\mathcal{Q}/\mathcal{L}$  is preserved.

Restricting to the case  $n = 2$ , it is then a simple matter to reduce the proof of Theorem 1 to the case  $\theta = 0$ . In particular, if  $M^3 \subset \mathbb{C}^2$  satisfies  $\mathcal{Q}_{r,p} = e^{i\theta} \mathcal{L}_{r,p}$  for  $\theta \in (0, 2\pi)$ , then the affine transformation  $F(z_1, z_2) = (z_1, e^{i\theta/2} z_2)$  determines a surface  $F(M)$  for which  $\mathcal{Q}_{r \circ F^{-1}, F(p)} = \mathcal{L}_{r \circ F^{-1}, F(p)}$ . If Theorem 1 holds for  $\theta = 0$ , it follows that  $F(M)$  is the image under an affine map  $G(w) = Aw + b$ , where  $0 \neq \det A \in \mathbb{R}$ , of a tubed surface over a complete strongly convex curve in the plane spanned by the  $\text{Re}(w_1)$  and  $\text{Re}(w_2)$  directions. Applying  $F^{-1}$ , it follows that  $M$  is the image under the affine map  $\tilde{G} = F^{-1} \circ G \circ F$  of a tubed surface over a complete strongly convex curve in the plane spanned by the  $\text{Re}(z_1)$  and  $\text{Re}(e^{i\theta/2} z_2)$  directions. If  $\tilde{G}$  is expressed as  $\tilde{G}(z) = \tilde{A}z + \tilde{b}$ , then  $\det \tilde{A} = \det A$ , so that  $0 \neq \det \tilde{A} \in \mathbb{R}$ , and the reduction is complete.

## 4. PROOF OF THEOREM 1

The proof of Theorem 1 uses the structural equations for a hypersurface in a manner similar to how they were used to prove the main results in [3, 4]. In this case, the equations are used to show that the rank of the second fundamental form of  $M$  is at most one. Since  $M$  is assumed to be complete, it follows from the result of Hartman and Nirenberg that  $M$  is cylindrical. After reconsidering condition (1) for cylindrical surfaces, it follows that  $M$  has the shape described in Theorem 1.

Let  $r$  be a defining function for  $M^3 \subset \mathbb{C}^2$  that is normalized so that  $|\nabla r| \equiv 2$ . From now on we use the preferred system of vectors represented in complex coordinates,

$$(2) \quad N = (r_{\bar{1}}, r_{\bar{2}}), \quad JN = (ir_{\bar{1}}, ir_{\bar{2}}), \quad X = (-r_2, r_1), \quad JX = (-ir_2, ir_1),$$

where subscripts and barred-subscripts refer to holomorphic and antiholomorphic partial derivatives, respectively. These vectors form an orthonormal basis of tangent vectors in  $\mathbb{C}^2$  along  $M$ .

With these choices, it follows from Proposition 3 in [4] that both

$$\begin{aligned} \mathcal{L}_{r,p} &= \frac{1}{2} (b(X, X) + b(JX, JX)), \\ \mathcal{Q}_{r,p} &= \frac{1}{2} (b(X, X) - b(JX, JX)) - \frac{i}{2} (b(X, JX) + b(JX, X)). \end{aligned}$$

Following the remark made in the final paragraph of the last section, we assume that  $\theta = 0$ . Then  $\mathcal{Q}_{r,p} = \mathcal{L}_{r,p}$ , and this implies that  $b(X, JX) = b(JX, X) = 0$ ,  $b(JX, JX) = 0$ , and  $b(X, X) = \lambda$  for a real function  $\lambda$  on  $M$ . The Levi non-degeneracy of  $M$  implies that  $\lambda$  is never zero.

The second fundamental form for  $M^3 \subset \mathbb{C}^2$  can then be represented by the  $3 \times 3$  matrix of real functions

$$\begin{pmatrix} \alpha & \beta & \gamma \\ \beta & \lambda & 0 \\ \gamma & 0 & 0 \end{pmatrix}.$$

The rows and columns correspond with tangent vectors  $JN$ ,  $X$ ,  $JX$ , respectively, as defined in (2). This means, in particular, that the Weingarten map is given by

$$\begin{aligned} S(JN) &= \alpha JN + \beta X + \gamma JX, \\ S(X) &= \beta JN + \lambda X, \\ S(JX) &= \gamma JN. \end{aligned}$$

The following lemma was proved in [3] and is useful for computing the connection along  $M$  expressed in terms of the system (2).

**Lemma 1.** *Let  $M^3 \subset \mathbb{C}^2$  be twice differentiable and have second fundamental form as described above. If  $Y \in TM$ , then  $\langle \bar{d}_Y X, JX \rangle = -\langle JN, \bar{d}_Y N \rangle$ . In particular,*

$$\begin{aligned} \langle \bar{d}_{JN} X, JX \rangle &= -\alpha, \\ \langle \bar{d}_X X, JX \rangle &= -\beta, \\ \langle \bar{d}_{JX} X, JX \rangle &= -\gamma. \end{aligned}$$

It is then a simple matter to give a full description of the connection  $\bar{d}$  in  $\mathbb{C}^2$  along  $M$  and, upon restriction to  $M$ , of the connection  $d$  on  $M$  as well.

**Lemma 2.** *Let  $M^3 \subset \mathbb{C}^2$  be twice differentiable and have second fundamental form as described above. Then the connection on  $M$  is given by*

$$\begin{aligned}d_{JN}JN &= -\gamma X + \beta JX, \\d_{JN}X &= +\gamma JN - \alpha JX, \\d_{JN}JX &= -\beta JN + \alpha X, \\d_XJN &= +\lambda JX, \\d_XX &= -\beta JX, \\d_XJX &= -\lambda JN + \beta X, \\d_{JX}JN &= 0, \\d_{JX}X &= -\gamma JX, \\d_{JX}JX &= +\gamma X.\end{aligned}$$

*Proof.* These identities are proved in the same way as for the corresponding identities in Lemmas 2 and 3 in [4]. The details are omitted.  $\square$

The structural equations then lead immediately to the conclusion that the rank of the second fundamental form of  $M$  is at most one.

**Lemma 3.** *Suppose  $M^3 \subset \mathbb{C}^2$  is three times differentiable and has second fundamental form as described above. Then  $\gamma = 0$  and  $\alpha\lambda - \beta^2 = 0$  on  $M$ .*

*Proof.* To show that  $\gamma = 0$  we apply the Codazzi equation to  $X, JX$ ,

$$\begin{aligned}0 &= d_XS(JX) - d_{JX}S(X) - S(d_XJX - d_{JX}X) \\&= d_X(\gamma JN) - d_{JX}(\beta JN + \lambda X) - S(-\lambda JN + \beta X + \gamma JX) \\&= X(\gamma)JN + \gamma \cdot \lambda JX - JX(\beta)JN - \beta \cdot 0 - JX(\lambda)X + \lambda \cdot \gamma JX \\&\quad + \lambda(\alpha JN + \beta X + \gamma JX) - \beta(\beta JN + \lambda X) - \gamma \cdot \gamma JN \\&= a_1JN + a_2X + a_3JX,\end{aligned}$$

where

$$\begin{aligned}a_1 &= X(\gamma) - JX(\beta) + \alpha\lambda - \beta^2 - \gamma^2, \\a_2 &= -JX(\lambda), \\a_3 &= +3\gamma\lambda.\end{aligned}$$

The requirement that  $a_3 = 0$  forces  $\gamma = 0$  since  $\lambda \neq 0$ . To show that  $\alpha\lambda - \beta^2 = 0$  we apply the Codazzi equation to  $X, JN$ , using  $\gamma = 0$  throughout,

$$\begin{aligned}0 &= d_XS(JN) - d_{JN}S(X) - S(d_XJN - d_{JN}X) \\&= d_X(\alpha JN + \beta X) - d_{JN}(\beta JN + \lambda X) - S((\alpha + \lambda)JX) \\&= X(\alpha)JN + \alpha \cdot \lambda JX + X(\beta)X - \beta \cdot \beta JX \\&\quad - JN(\beta)JN - \beta \cdot \beta JX - JN(\lambda)X + \lambda \cdot \alpha JX - (\alpha + \lambda) \cdot 0 \\&= a_1JN + a_2X + a_3JX,\end{aligned}$$

where

$$\begin{aligned}a_1 &= X(\alpha) - JN(\beta), \\a_2 &= X(\beta) - JN(\lambda), \\a_3 &= 2\alpha\lambda - 2\beta^2.\end{aligned}$$

Again, the requirement that  $a_3 = 0$  forces  $\alpha\lambda - \beta^2 = 0$ .  $\square$

It follows from Lemma 3 that the rank of the second fundamental form of  $M$  is at most one; that is, the curvature of  $M$  is zero. Since  $M$  is assumed to be complete, it then follows from the theorem of Hartman and Nirenberg that  $M$  is a cylinder [6]. (See also Sternberg [8, p.269].)

With this information, we localize the surface by choosing a point  $p \in M$ , and following a translation in  $\mathbb{C}^2$ , we assume that  $p$  is the origin. By using a special unitary transformation (such a transformation is affine with real determinant), we also assume that the unit normal vector at  $p$  is  $(1, 0)$ . Then  $M$  can be defined near  $p$  by the equation

$$0 = z_1 + \bar{z}_1 + \operatorname{Re} \left( -\alpha_0 z_1^2 - 2i\beta_0 z_1 z_2 + \frac{\lambda_0}{2} z_2^2 \right) + \frac{\lambda_0}{2} z_2 \bar{z}_2 + o(|z|^2),$$

where  $\alpha_0, \beta_0$ , and  $\lambda_0$  are real constants with  $\alpha_0\lambda_0 - \beta_0^2 = 0$  and  $\lambda_0 \neq 0$ . These constants agree with the corresponding entries in the second fundamental form at the origin.

Finally, by using the origin-preserving affine map  $w = F(z) = (z_1, z_2 - i\beta_0 z_1/\lambda_0)$  and multiplying the defining equation by the function

$$h(w) = 1 - \frac{\beta_0^2}{4\lambda_0}(w_1 + \bar{w}_1) + \frac{i\beta_0}{2}(w_2 - \bar{w}_2),$$

which is positive near  $p$ , we obtain a new surface  $M' = F(M)$  that can be defined near the origin by

$$0 = w_1 + \bar{w}_1 + \lambda_0 (\operatorname{Re} w_2)^2 + o(|w|^2).$$

(Here,  $M$  is the original surface and  $F$  is an affine map with real determinant.) As the affine image of a cylinder,  $M'$  is also a cylinder, and therefore  $M'$  can be defined locally by

$$0 = w_1 + \bar{w}_1 + \psi(\operatorname{Re} w_2)$$

for some three times differentiable function  $\psi : \mathbb{R} \rightarrow \mathbb{R}$ .

By piecing together such local representations it is apparent that, even globally, the original surface  $M$  is the image under  $F^{-1}$  of the surface  $M'$  whose defining equation is independent of the variables  $\operatorname{Im} w_1$  or  $\operatorname{Im} w_2$ . So there exists a three times differentiable function  $\tilde{r} : \mathbb{R}^2 \rightarrow \mathbb{R}$  such that

$$M' = \{(w_1, w_2) \in \mathbb{C}^2 : \tilde{r}(\operatorname{Re} w_1, \operatorname{Re} w_2) = 0\},$$

where  $\nabla \tilde{r} \neq 0$  on the set  $\{\tilde{r} = 0\}$ . The condition that  $M$  is Levi non-degenerate means, too, that  $M'$  is Levi non-degenerate, and this corresponds exactly with the requirement that  $\{\tilde{r} = 0\}$  is a strongly convex curve in  $\mathbb{R}^2$ . Since  $M = F^{-1}(M')$ , and since affine maps preserve convexity, the conclusion of Theorem 1 follows.

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