12-1-2013

Instability indices for matrix polynomials

Todd Kapitula
Calvin University

Elizabeth Hibma
Calvin University

Hwa Pyeong Kim
Calvin University

Jonathan Timkovich
University of Michigan, Ann Arbor

Linear Algebra and its Applications 439 (2013) 3412–3434

Contents lists available at ScienceDirect

Linear Algebra and its Applications

www.elsevier.com/locate/laa

Instability indices for matrix polynomials

Todd Kapitula [a,*], Elizabeth Hibma [a], Hwa-Pyeong Kim [a], Jonathan Timkovich [b]

[a] Department of Mathematics and Statistics, Calvin College, Grand Rapids, MI 49546, United States
[b] Department of Mathematics, University of Michigan, Ann Arbor, MI 48109-1043, United States

ARTICLE INFO

Article history:
Received 24 October 2012
Accepted 26 August 2013
Available online 26 September 2013
Submitted by C. Mehl

MSC:
15A18
15A22
15B57

Keywords:
$\star$-even matrix polynomial
Hermitian matrix polynomial
Hamiltonian–Krein index

ABSTRACT

There is a well-established instability index theory for linear and quadratic matrix polynomials for which the coefficient matrices are Hermitian and skew-Hermitian. This theory relates the number of negative directions for the matrix coefficients which are Hermitian to the total number of unstable eigenvalues for the polynomial. Herein we extend the theory to $\star$-even matrix polynomials of any finite degree. In particular, unlike previously known cases we show that the instability index depends upon the size of the matrices when the degree of the polynomial is greater than two. We also consider Hermitian matrix polynomials, and derive an index which counts the number of eigenvalues with nonpositive imaginary part. The results are refined if we consider the Hermitian matrix polynomial to be a perturbation of a $\star$-even polynomials; however, this refinement requires additional assumptions on the matrix coefficients.

1. Introduction

We are interested in studying the structure of the spectrum for matrix polynomials of the form

$$\boldsymbol{P}_d(\lambda) := \sum_{j=0}^{d} \lambda^j \boldsymbol{M}_j, \tag{1.1}$$

where the matrices $\boldsymbol{M}_j \in \mathbb{C}^{n\times n}$ are:

* Corresponding author.
E-mail addresses: tmk5@calvin.edu (T. Kapitula), ekh7@students.calvin.edu (E. Hibma), hk52@students.calvin.edu (H.-P. Kim), iketimko@umich.edu (J. Timkovich).

0024-3795/$ – see front matter
http://dx.doi.org/10.1016/j.laa.2013.08.034

(a) alternating, i.e., $\boldsymbol{M}_{2i}$ is Hermitian and $\boldsymbol{M}_{2i+1}$ is skew-Hermitian;
(b) all Hermitian.

In the literature a matrix polynomial satisfying (a) is often referred to as a $\star$-even matrix polynomial [28,27], and we will adopt this nomenclature in this paper. We will call a polynomial satisfying case (b) a Hermitian matrix polynomial. Values of λ for which $\boldsymbol{P}_d(\lambda)$ is singular will be denoted here as polynomial eigenvalues. It will be assumed throughout that the matrices $\boldsymbol{M}_0$ and $\boldsymbol{M}_d$ are nonsingular. The case that $\boldsymbol{M}_0$ is singular was discussed in Bronski et al. [5] for $\star$-even quadratic polynomials, and the technique used in that paper to deal with the singularity can easily be applied to $\star$-even polynomials of higher degree. For both problems we are interested in determining the exact number of unstable polynomial eigenvalues, i.e., those values of λ with $\operatorname{Re}\lambda > 0$. In order to achieve an exact result, we will have to expand our notion of unstable polynomial eigenvalues to include those purely imaginary eigenvalues which have negative Krein index (signature).

The study of the index problem for linear pencils goes back (at least) to the work of Pontryagin [33]. He introduced the idea of studying the problem via a study of Hermitian matrices acting on Hilbert spaces with an indefinite inner product (Pontryagin spaces). Other referential works which study spectral problems with this perspective include Azizov and Iokhvidov [4], Gohberg et al. [9], and Shkalikov [34,35]. A different approach to studying the problem, based upon the techniques of spectral decomposition and simultaneous diagonalization of quadratic forms, is found in Chugunova and Pelinovsky [7], Hărăguş and Kapitula [14], and Pelinovsky [32].

A general study of matrix and operator polynomials is given in, e.g., Gohberg et al. [10], Markus [30]. A review of the current understanding regarding the number for unstable polynomial eigenvalues of linear $\star$-even polynomials is given in Section 2. Additional results include the following. Let $\boldsymbol{x}\cdot\boldsymbol{y}$ represent the standard inner product on $\mathbb{C}^n$. When the matrices are all Hermitian, it is known that if $\boldsymbol{P}_d(\mu)$ is definite for some $\mu\in\mathbb{R}$ (e.g., $\boldsymbol{M}_d$ or $\boldsymbol{M}_0$ is definite), and if the dth-order polynomial $p_{\boldsymbol{x}}(\lambda)=\boldsymbol{x}\cdot\boldsymbol{P}_d(\lambda)\boldsymbol{x}$ has d distinct real zeros for each nonzero $\boldsymbol{x}\in\mathbb{C}^n$, then all of the polynomial eigenvalues are real-valued and algebraically simple [2, Theorem 3.8] (also see [1]). The definiteness condition on the coefficients can be relaxed to simply assuming that the roots of $p_{\boldsymbol{x}}(\lambda)$ are distinct and real-valued for any $\boldsymbol{x}$ [2, Theorem 4.7]. In both cases, it is not clear what is the number of unstable polynomial eigenvalues. If the polynomial is $\star$-even, then the observation that $\mathrm{i}\boldsymbol{M}_{2i+1}$ is Hermitian for each i yields that if $p_{\boldsymbol{x}}(\mathrm{i}\lambda)$ has d distinct real zeros for all $\boldsymbol{x}\in\mathbb{C}^n$, then all of the spectrum will be purely imaginary and algebraically simple. Unfortunately, this stability criteria is not easily verified. Henrion et al. [12] considered the more general problem of formulating conditions which ensured that the polynomial eigenvalues belonged to a given region of the complex plane. For a given region $D\subset\mathbb{C}$ it was shown in [12, Section 4] that if the optimal value μ of the optimization problem

$$\mu=\min_{\lambda\in D^c}\boldsymbol{P}_d(\lambda)\boldsymbol{x}\cdot\boldsymbol{P}_d(\lambda)\boldsymbol{x}$$

is positive for any nonzero $\boldsymbol{x}\in\mathbb{C}^n$, then all of the polynomial eigenvalues belong to D. Furthermore, the optimization problem was shown to be solvable via efficient interior point methods by recasting the problem as a Linear Matrix Inequality. Upper and lower bounds for the absolute values of the polynomial eigenvalues are derived by Higham and Tisseur [13]. In particular, after setting

$$r:=\min\bigl\{\|\boldsymbol{M}_j\|^{-1/j}\colon\ j=1,\dots,d\bigr\},\qquad R:=\max\bigl\{\|\boldsymbol{M}_j\|^{1/(d-j)}\colon\ j=0,\dots,d-1\bigr\},$$

the polynomial eigenvalues satisfy the bounds

$$\frac{r}{1+\|\boldsymbol{M}_0^{-1}\|}\leqslant|\lambda|\leqslant R\bigl(1+\|\boldsymbol{M}_d^{-1}\|\bigr).$$

Note that $\boldsymbol{M}_0,\boldsymbol{M}_d$ being nonsingular guarantees that all of the polynomial eigenvalues are contained within some finite annulus in the complex plane.

If we assume that each $\boldsymbol{M}_j:=m_j\in\mathbb{R}^{1\times1}$, i.e., the polynomial is actually a dth-order polynomial with real-valued coefficients, then the Routh–Hurwitz stability criterion can be used to determine the exact number of zeros with positive real part. The extension to matrix-valued polynomials was reported in [23]. If in the scalar case all of the coefficients are (without loss of generality) positive, then it will be the case that there are no real-valued and positive zeros. This result holds for Hermitian polynomials if all of the matrix coefficients are positive definite. In addition, if for the scalar problem all of the coefficients in the first column of the Routh array are also positive, then all of the zeros must satisfy $\operatorname{Re}\lambda < 0$. If either of these conditions are violated, there will necessarily be some number of zeros with nonnegative real part. For example, if we assume that the polynomial is monic, i.e., $m_d = 1$, then the condition for all eigenvalues to have negative real part becomes:

$$d = 2: \quad m_1, m_0 > 0,$$
$$d = 3: \quad m_2, m_0 > 0,\ m_1 > m_0/m_2,$$
$$d = 4: \quad m_3, m_1, m_0 > 0,\ m_2 > \left(m_1^2 + m_0 m_3^2\right)/(m_1 m_3).$$

The extension of these results to polynomials with complex-valued coefficients was reported in [8]. Unfortunately, it does not appear to be the case that such precise stability criterion results are available for Hermitian polynomials. However, it is shown in Section 4.4 that for a singularly perturbed cubic Hermitian polynomial,

$$\boldsymbol{P}_3^{\epsilon}(\lambda) = \boldsymbol{M}_0 + \lambda\boldsymbol{M}_1 + \lambda^2\boldsymbol{M}_2 + \epsilon\lambda^3\boldsymbol{M}_3, \quad 0 < \epsilon \ll 1,$$

if $\boldsymbol{M}_1, \boldsymbol{M}_2 > 0$ (i.e., are positive definite), then there are precisely $\mathrm{n}(\boldsymbol{M}_0) + \mathrm{n}(\boldsymbol{M}_3)$ unstable and real-valued polynomial eigenvalues. Here $\mathrm{n}(\boldsymbol{S})$ denotes the number of negative eigenvalues (counting multiplicity) for the Hermitian matrix $\boldsymbol{S}$. In particular, for the singularly perturbed cubic Hermitian polynomial the coefficients all being positive definite implies that there are no unstable polynomial eigenvalues. As we will see via numerical simulation, this result crucially depends upon the singular nature of the problem.

The paper is organized as follows. In Section 2 we review the Hamiltonian–Krein (instability) index theory for linear $\star$-even polynomials. In Section 3 we extend the theory to higher-order $\star$-even polynomials. Finally, in Section 4 we derive an index theory for Hermitian polynomials. We first count the number of eigenvalues with nonpositive imaginary part. Afterwards, we take a perturbative approach to consider the unstable polynomial eigenvalue problem. Additional assumptions are needed on the matrix coefficients, so the results for this problem are not as general as those given in Section 3.

2. Hamiltonian–Krein index theory for $\star$-even linear polynomials: A review

Here we will consider $\star$-even linear polynomials of the form

$$\boldsymbol{P}_1(\lambda) = \boldsymbol{S} + \lambda\boldsymbol{J}, \tag{2.1}$$

where $\boldsymbol{S} \in \mathbb{C}^{n\times n}$ is Hermitian and $\boldsymbol{J} \in \mathbb{C}^{n\times n}$ is skew-Hermitian. Both matrices are assumed to be invertible. A longer discussion on this material can be found in, e.g., [18, Chapter 7.1], for the case that both matrices have real-valued entries. The infinite-dimensional version of the case discussed herein can be found in Hărăguş and Kapitula [14]. The interested reader may also wish to consult Chugunova and Pelinovsky [7], Pelinovsky [32] for a different approach to the problem.

The polynomial eigenvalues for (2.1) are symmetric with respect to the imaginary axis, i.e., they come in the pairs $\{\lambda, -\bar{\lambda}\}$. If the matrices $\boldsymbol{S}$, $\boldsymbol{J}$ have real-valued entries only, or if $n = 2\ell$ is even and the matrices have the (canonical) form

$$\boldsymbol{J} = \begin{pmatrix} \boldsymbol{0}_\ell & -\boldsymbol{I}_\ell \\ \boldsymbol{I}_\ell & \boldsymbol{0}_\ell \end{pmatrix}, \qquad \boldsymbol{S} = \begin{pmatrix} \boldsymbol{S}_+ & \boldsymbol{0}_\ell \\ \boldsymbol{0}_\ell & \boldsymbol{S}_- \end{pmatrix},$$

where each $\boldsymbol{S}_\pm \in \mathbb{C}^{\ell\times\ell}$ is Hermitian, then the polynomial eigenvalues have the quartet symmetry $\{\pm\lambda, \pm\bar{\lambda}\}$. Here $\boldsymbol{I}_\ell \in \mathbb{R}^{\ell\times\ell}$ is the identity matrix, and $\boldsymbol{0}_\ell \in \mathbb{R}^{\ell\times\ell}$ is the zero matrix. Let k_r denote the number of positive real-valued polynomial eigenvalues (counting multiplicity), and k_c the number of complex-valued polynomial eigenvalues with positive real part (counting multiplicity). If the polynomial eigenvalues satisfy the quartet symmetry, k_c will be an even integer. If $\lambda = \mathrm{i}\lambda_0 \in \mathrm{i}\mathbb{R}$ is a polynomial eigenvalue with associated generalized eigenspace $\mathbb{E}_{\mathrm{i}\lambda_0}$, i.e., $\boldsymbol{J}^{-1}\boldsymbol{S} : \mathbb{E}_{\mathrm{i}\lambda_0} \mapsto \mathbb{E}_{\mathrm{i}\lambda_0}$ with $\sigma(\boldsymbol{J}^{-1}\boldsymbol{S}|_{\mathbb{E}_{\mathrm{i}\lambda_0}}) = \{\mathrm{i}\lambda_0\}$, denote the negative Krein index of that polynomial eigenvalue by

$$k_\mathrm{i}^-(\mathrm{i}\lambda_0) = \mathrm{n}(\boldsymbol{S}|_{\mathbb{E}_{\mathrm{i}\lambda_0}}). \tag{2.2}$$

The positive Krein index is given by

$$k_\mathrm{i}^+(\mathrm{i}\lambda_0) = \mathrm{p}(\boldsymbol{S}|_{\mathbb{E}_{\mathrm{i}\lambda_0}}) = \dim[\mathbb{E}_{\mathrm{i}\lambda_0}] - k_\mathrm{i}^-(\mathrm{i}\lambda_0).$$

The total negative (positive) Krein index is the sums of all the individual indices,

$$k_\mathrm{i}^\pm = \sum k_\mathrm{i}^\pm(\mathrm{i}\lambda_0).$$

If the polynomial eigenvalues satisfy the quartet symmetry it will be true that $k_\mathrm{i}^\pm(-\mathrm{i}\lambda_0) = k_\mathrm{i}^\pm(\mathrm{i}\lambda_0)$; hence, as is the case for k_c it will be true that $k_\mathrm{i}^\pm$ is an even integer. The negative (positive) Hamiltonian–Krein index is the sum of these individual indices, i.e.,

$$K_{\mathrm{Ham}}^\pm := k_\mathrm{r} + k_\mathrm{c} + k_\mathrm{i}^\pm. \tag{2.3}$$

In addition to being a useful counter within the Hamiltonian–Krein index, the Krein index of a polynomial eigenvalue has dynamical implications (e.g., see Krein [24,25], Krein and Kjubarskii [26], or the discussion in [17, Chapter 7.1.2]). Suppose that $\lambda_1, \lambda_2 \in \mathrm{i}\mathbb{R}$ are simple polynomial eigenvalues, and further suppose that $\boldsymbol{S} = \boldsymbol{S}(\epsilon)$ is smooth for some parameter ϵ. As long as $\lambda_j = \lambda_j(\epsilon)$ are simple, they will also be smooth in ϵ, and will remain purely imaginary. Now suppose for a critical value of ϵ, say ϵ_0, the two eigenvalues collide, so that $\lambda_1(\epsilon_0)$ is an eigenvalue with algebraic multiplicity two. If for $\epsilon \neq \epsilon_0$ the two eigenvalues have the same Krein index,[1] then $\lambda_1(\epsilon_0)$ will be algebraically simple. Furthermore, it will be the case that the polynomial eigenvalues will remain purely imaginary for all ϵ in a neighborhood of ϵ_0. On the other hand, suppose that the polynomial eigenvalues have opposite Krein index for $\epsilon \neq \epsilon_0$. It will then generically be the case that

(a) $\lambda_1(\epsilon_0)$ will be a polynomial eigenvalue with geometric multiplicity one and algebraic multiplicity two;
(b) for either $\epsilon < \epsilon_0$ or $\epsilon > \epsilon_0$ the polynomial eigenvalues will be complex conjugates with nonzero real part.

In other words, $\epsilon = \epsilon_0$ is the Hamiltonian–Hopf bifurcation point. The situation is graphically depicted in Fig. 1.

The major result is that the Hamiltonian–Krein indices for linear $\star$-even polynomials are related to the negative (positive) index of $\boldsymbol{S}$. The focus of Section 3 is to extend this result to $\star$-even matrix polynomials of arbitrary finite degree.

Lemma 2.1. *The Hamiltonian–Krein indices* (2.3) *for the linear* $\star$*-even polynomial* (2.1) *satisfy*

$$K_{\mathrm{Ham}}^- = \mathrm{n}(\boldsymbol{S}), \qquad K_{\mathrm{Ham}}^+ = \mathrm{p}(\boldsymbol{S}).$$

Remark 2.2. In general, the result of Lemma 2.1 is stated only for the negative Hamiltonian–Krein index. The result for the positive index follows immediately from the fact that the eigenvalue symmetry implies

$$K_{\mathrm{Ham}}^- + K_{\mathrm{Ham}}^+ = n.$$

[1] As long as the polynomial eigenvalue is simple, its Krein index as a function of ϵ will not change.

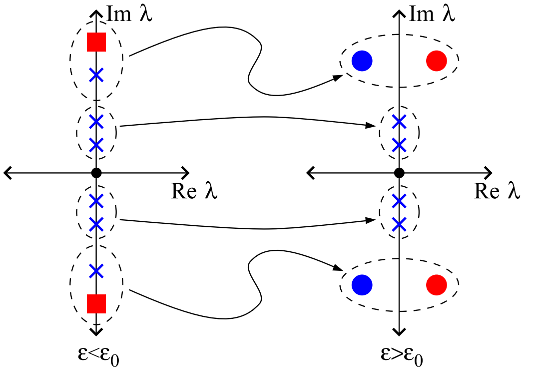

Fig. 1. (Color online.) A cartoon showing the collision of simple polynomial eigenvalues on the imaginary axis. In the left panel there are four pairs of simple purely imaginary polynomial eigenvalues: one pair has a negative Krein index of one ((red) squares), and the other three pairs have a negative Krein index of zero ((blue) crosses). The dotted oval denotes the pairs of polynomial eigenvalues which collide when $\epsilon = \epsilon_0$. The collision of polynomial eigenvalues with opposite sign (generically) leads to a quartet of polynomial eigenvalues with nonzero real and imaginary parts. If the polynomial eigenvalues have the same Krein index, they will remain simple and stay on the imaginary axis.

Remark 2.3. The Hamiltonian–Krein index at one level has the following interpretation. The Hermitian matrix $\boldsymbol{S}$ determines the total number of (potentially) unstable polynomial eigenvalues, while the skew-Hermitian matrix $\boldsymbol{J}$ determines the location of these polynomial eigenvalues.

We can refine the result of Lemma 2.1 in the canonical case where

$$\boldsymbol{J} = \begin{pmatrix} \boldsymbol{0}_\ell & -\boldsymbol{I}_\ell \\ \boldsymbol{I}_\ell & \boldsymbol{0}_\ell \end{pmatrix}, \qquad \boldsymbol{S} = \begin{pmatrix} \boldsymbol{S}_+ & \boldsymbol{0}_\ell \\ \boldsymbol{0}_\ell & \boldsymbol{S}_- \end{pmatrix}.$$

Recall that now the polynomial eigenvalues satisfy the quartet symmetry, with k_c and $k_i^\pm$ being even integers. In this case finding the spectrum of the linear polynomial is equivalent to finding that for the quadratic Hermitian polynomial

$$\boldsymbol{P}_2(\lambda) = \boldsymbol{S}_+ + \lambda^2 \boldsymbol{S}_-^{-1}.$$

For this quadratic polynomial there is a lower bound on the number of positive real-valued polynomial eigenvalues; namely, $k_r \geqslant |\mathrm{n}(\boldsymbol{S}_+) - \mathrm{n}(\boldsymbol{S}_-)|$ (see Grillakis [11], Jones [15], Jones and Moloney [16], Kapitula and Promislow [17] and Lemma 4.1 for a generalization to Hermitian polynomials of arbitrary finite degree). Under the additional assumption that all of the real-valued polynomial eigenvalues are algebraically simple, we can precisely determine the number of (potentially) unstable polynomial eigenvalues with nonzero imaginary part through the intersection of two negative cones. For a Hermitian matrix $\boldsymbol{S}$, the negative cone is given by

$$C^-(\boldsymbol{S}) = \{\boldsymbol{x}\colon\ \boldsymbol{S}\boldsymbol{x}\cdot\boldsymbol{x} < 0\} \cup \{\boldsymbol{0}\},$$

and $\dim[C^-(\boldsymbol{S})]$ is the dimension of a maximal subspace contained in $C^-(\boldsymbol{S})$. The maximal subspace is not unique, but the dimension is unique, and is given by $\dim[C^-(\boldsymbol{S})] = \mathrm{n}(\boldsymbol{S})$. If all of the real polynomial eigenvalues are algebraically simple, then it is the case that the polynomial eigenvalues with nonzero imaginary part satisfy

$$k_c + k_i^- = 2\dim\left[C^-(\boldsymbol{S}_+) \cap C^-\left(\boldsymbol{S}_-^{-1}\right)\right],$$

so that

$$k_r = \left|\mathrm{n}(\boldsymbol{S}_+) - \mathrm{n}(\boldsymbol{S}_-)\right| + 2\left(\min\left\{\mathrm{n}(\boldsymbol{S}_+), \mathrm{n}(\boldsymbol{S}_-)\right\} - \dim\left[C^-(\boldsymbol{S}_+) \cap C^-\left(\boldsymbol{S}_-^{-1}\right)\right]\right)$$

[14, Corollary 2.26]. If this simplicity assumption on the purely real polynomial eigenvalues is removed, then

$$k_c + k_i^- \leqslant 2\dim\left[C^-(\boldsymbol{S}_+) \cap C^-\left(\boldsymbol{S}_-^{-1}\right)\right],$$

and there is the corresponding lower bound for k_r,

$$k_r \geqslant \left|n(\boldsymbol{S}_+) - n(\boldsymbol{S}_-)\right| + 2\left(\min\left\{n(\boldsymbol{S}_+), n(\boldsymbol{S}_-)\right\} - \dim\left[C^-(\boldsymbol{S}_+) \cap C^-\left(\boldsymbol{S}_-^{-1}\right)\right]\right).$$

While we will not discuss it here, a lower bound on the total number of these polynomial eigenvalues with nonzero imaginary part can also be derived [14, Remark 2.27].

Finally, under the assumption that the Hermitian matrices satisfy for all nonzero $\boldsymbol{x}$ the estimate

$$\boldsymbol{S}_+\boldsymbol{x}\cdot\boldsymbol{x} \geqslant \alpha\boldsymbol{S}_-\boldsymbol{x}\cdot\boldsymbol{x}, \quad \alpha \neq 0,$$

then

$$k_r = \begin{cases} |n(\boldsymbol{S}_+) - n(\boldsymbol{S}_-)|, & \alpha > 0, \\ n(\boldsymbol{S}_+) + n(\boldsymbol{S}_-), & \alpha < 0, \end{cases}$$

see Azizov and Chugunova [3]. Consequently, when $\alpha < 0$ all of the unstable polynomial eigenvalues must be real-valued, whereas if $\alpha > 0$ there will be precisely $2\min\{n(\boldsymbol{S}_+), n(\boldsymbol{S}_-)\}$ (potentially) unstable polynomial eigenvalues with nonzero imaginary part. In other words, when $\alpha < 0$ the negative cones have a trivial intersection, whereas if $\alpha > 0$ the dimension of the subspace contained in the intersection of the respective negative cones is maximal.

3. Hamiltonian–Krein index for ⋆-even polynomials

The key to determining the Hamiltonian–Krein index for ⋆-even polynomials of arbitrary finite degree lies in finding a linearization that respects the Hamiltonian structure of the polynomial eigenvalue problem. This was accomplished by Mehrmann and Watkins [31]. Therein they showed that the dth-order polynomial eigenvalue problem (1.1) is equivalent to the linear problem (2.1) with

$$\boldsymbol{S} = \left(\begin{array}{c|cccccc} \boldsymbol{M}_0 & \boldsymbol{0}_n & \boldsymbol{0}_n & \boldsymbol{0}_n & \cdots & \boldsymbol{0}_n \\ \hline \boldsymbol{0}_n & \boldsymbol{M}_2 & \boldsymbol{M}_3 & \boldsymbol{M}_4 & \cdots & \boldsymbol{M}_d \\ \boldsymbol{0}_n & -\boldsymbol{M}_3 & -\boldsymbol{M}_4 & & & \boldsymbol{0}_n \\ \boldsymbol{0}_n & \boldsymbol{M}_4 & & & & \boldsymbol{0}_n \\ \vdots & \vdots & & & & \vdots \\ \boldsymbol{0}_n & \mp\boldsymbol{M}_d & \boldsymbol{0}_n & \boldsymbol{0}_n & \cdots & \boldsymbol{0}_n \end{array}\right)$$

and

$$\boldsymbol{J} = \left(\begin{array}{ccccc|c} \boldsymbol{M}_1 & \boldsymbol{M}_2 & \boldsymbol{M}_3 & \cdots & \boldsymbol{M}_{d-1} & \boldsymbol{M}_d \\ \hline -\boldsymbol{M}_2 & -\boldsymbol{M}_3 & -\boldsymbol{M}_4 & \cdots & -\boldsymbol{M}_d & \boldsymbol{0}_n \\ \boldsymbol{M}_3 & \boldsymbol{M}_4 & & & \boldsymbol{0}_n & \boldsymbol{0}_n \\ -\boldsymbol{M}_4 & & & & \boldsymbol{0}_n & \boldsymbol{0}_n \\ \vdots & & & & \vdots & \vdots \\ \pm\boldsymbol{M}_d & \boldsymbol{0}_n & \boldsymbol{0}_n & \cdots & \boldsymbol{0}_n & \boldsymbol{0}_n \end{array}\right).$$

Here $\pm\boldsymbol{M}_d$ represents $(-1)^{d-1}\boldsymbol{M}_d$. As long as the matrices $\boldsymbol{M}_0$ and $\boldsymbol{M}_d$ are nonsingular, so are the block matrices $\boldsymbol{J}$ and $\boldsymbol{S}$. It is also clear that $\boldsymbol{S}$ is Hermitian and $\boldsymbol{J}$ is skew-Hermitian. The associated eigenvectors are related through $\boldsymbol{P}_d(\lambda)\boldsymbol{x} = \boldsymbol{0}$ if and only if $(\boldsymbol{S} + \lambda\boldsymbol{J})\boldsymbol{v} = \boldsymbol{0}$ with $\boldsymbol{v} = (\boldsymbol{x}, \lambda\boldsymbol{x}, \lambda^2\boldsymbol{x}, \ldots, \lambda^{d-1}\boldsymbol{x})^{\mathrm{T}}$. Through this linearization we immediately see that the polynomial eigenvalues for the ⋆-even polynomial of higher degree satisfies the spectral symmetry $\{\lambda, -\bar{\lambda}\}$ of the linear problem (see Mackey et al. [29,27,28] for an alternate proof).

3.1. General result

In order to apply the results of Section 2 to the linearized system, we must do two things: (a) compute $\mathrm{n}(\boldsymbol{S})$, and (b) relate the negative Krein index for the linearization to that for the original polynomial. In order to compute $\mathrm{n}(\boldsymbol{S})$, we find the following result to be most useful:

Lemma 3.1. *Let $\boldsymbol{S} \in \mathbb{C}^{2m\times 2m}$ be Hermitian and have the block form*

$$\boldsymbol{S} = \begin{pmatrix} \boldsymbol{S}_1 & \boldsymbol{S}_2 \\ \boldsymbol{S}_2^{\mathrm{H}} & \boldsymbol{0}_m \end{pmatrix},$$

where $\boldsymbol{S}_1 \in \mathbb{C}^{m\times m}$ is Hermitian. Assuming that $\boldsymbol{S}_2$ is nonsingular,

$$\mathrm{n}(\boldsymbol{S}) = m.$$

Proof. The eigenvalues for $\boldsymbol{S}$ are found by solving

$$\boldsymbol{S}_1\boldsymbol{x} + \boldsymbol{S}_2\boldsymbol{y} = \mu\boldsymbol{x}, \qquad \boldsymbol{S}_2^{\mathrm{H}}\boldsymbol{x} = \mu\boldsymbol{y},$$

which is equivalent to solving the quadratic Hermitian polynomial

$$\boldsymbol{P}_2(\mu)\boldsymbol{x} = \boldsymbol{0}, \qquad \boldsymbol{P}_2(\mu) = \mu^2\boldsymbol{I}_m - \mu\boldsymbol{S}_1 - \boldsymbol{S}_2\boldsymbol{S}_2^{\mathrm{H}}.$$

We know that for fixed $\mu \in \mathbb{R}$ the polynomial eigenvalues for $\boldsymbol{P}_2(\mu)$ must be real-valued. Since the polynomial itself is Hermitian for real-valued μ, there exist m real analytic eigenvalues $r_j(\mu)$ and associated real analytic eigenvectors $\boldsymbol{z}_j(\mu)$ which solve $\boldsymbol{P}_2(\mu)\boldsymbol{z}_j(\mu) = r_j(\mu)\boldsymbol{z}_j(\mu)$. For large $|\mu|$ we have

$$\frac{\boldsymbol{P}_2(\mu)}{\mu^2} = \boldsymbol{I}_m + \mathcal{O}\big(|\mu|^{-1}\big);$$

hence, for large $|\mu|$ each of these eigenvalues has the asymptotics

$$r_j(\mu) = \mu^2 + \mathcal{O}\big(|\mu|\big) > 0, \quad |\mu| \gg 1.$$

Since $\boldsymbol{P}_2(0) = -\boldsymbol{S}_2\boldsymbol{S}_2^{\mathrm{H}}$, it must further be the case that $r_j(0) \in \sigma(-\boldsymbol{S}_2\boldsymbol{S}_2^{\mathrm{H}})$ for each j. Since $\boldsymbol{S}_2$ is nonsingular, $\boldsymbol{S}_2\boldsymbol{S}_2^{\mathrm{H}}$ is a positive-definite matrix. This implies that $r_j(0) < 0$ for each j. By continuity we then have that $r_j(\mu) = 0$ has at least one positive and at least one negative solution for each j. Since $\det[\boldsymbol{P}_2(\mu)] = 0$ if and only if $r_j(\mu) = 0$ for some j, and since $\det[\boldsymbol{P}_2(\mu)]$ is a polynomial in μ of degree $2m$, we conclude that $\det[\boldsymbol{P}_2(\mu)]$ has precisely m positive zeros and m negative zeros. In conclusion, $\mathrm{n}(\boldsymbol{S}) = m$. □

Remark 3.2. A more general result concerning the number of negative directions for block matrices is available via Kostenko [22, Lemma 1]. If $\boldsymbol{S} \in \mathbb{C}^{2m\times 2m}$ is Hermitian and has the block form

$$\boldsymbol{S} = \begin{pmatrix} \boldsymbol{S}_1 & \boldsymbol{S}_2 \\ \boldsymbol{S}_2^{\mathrm{H}} & \boldsymbol{S}_3 \end{pmatrix},$$

where $\boldsymbol{S}_1, \boldsymbol{S}_3 \in \mathbb{C}^{m\times m}$ are Hermitian, then assuming that $\boldsymbol{S}_1$ is nonsingular,

$$\mathrm{n}(\boldsymbol{S}) = \mathrm{n}(\boldsymbol{S}_1) + \mathrm{n}\big(\boldsymbol{S}_3 - \boldsymbol{S}_2^{\mathrm{H}}\boldsymbol{S}_1^{-1}\boldsymbol{S}_2\big).$$

Using Lemma 3.1, we have the following result concerning the Hermitian linearization matrix $\boldsymbol{S}$:

Lemma 3.3. *Suppose that $\boldsymbol{M}_0$ and $\boldsymbol{M}_d$ are invertible. If $d = 2\ell - 1$ for $\ell \in \mathbb{N}$, then*

$$\mathrm{n}(\boldsymbol{S}) = \mathrm{n}(\boldsymbol{M}_0) + (\ell - 1)n, \qquad \mathrm{p}(\boldsymbol{S}) = \mathrm{p}(\boldsymbol{M}_0) + (\ell - 1)n,$$

while if $d = 2\ell$ *for* $\ell \in \mathbb{N}$*, then*

$$\mathrm{n}(\boldsymbol{S}) = \mathrm{n}(\boldsymbol{M}_0) + \mathrm{n}\big((-1)^{\ell-1}\boldsymbol{M}_{2\ell}\big) + (\ell-1)n,$$
$$\mathrm{p}(\boldsymbol{S}) = \mathrm{p}(\boldsymbol{M}_0) + \mathrm{p}\big((-1)^{\ell-1}\boldsymbol{M}_{2\ell}\big) + (\ell-1)n.$$

Proof. First suppose that $d = 2\ell - 1$. It is clear that

$$\mathrm{n}(\boldsymbol{S}) = \mathrm{n}(\boldsymbol{M}_0) + \mathrm{n}(\boldsymbol{S}_{\mathrm{red}}),$$

where $\boldsymbol{S}_{\mathrm{red}} \in \mathbb{C}^{(2\ell-2)n\times(2\ell-2)n}$ is the lower right-hand block. Further examination of $\boldsymbol{S}_{\mathrm{red}}$ shows that it is of the block form described in Lemma 3.1 with $\boldsymbol{S}_3 = \boldsymbol{0}_{(\ell-1)n}$, and with the invertibility of the matrix $\boldsymbol{S}_2$ being ensured by the invertibility of $\boldsymbol{M}_{2\ell-1}$. The result for the negative index now follows from that lemma. The positive index follows immediately from the facts that $\mathrm{p}(\boldsymbol{S}) = (2\ell-1)n - \mathrm{n}(\boldsymbol{S})$ and $\mathrm{p}(\boldsymbol{M}_0) = n - \mathrm{n}(\boldsymbol{M}_0)$.

Now suppose that $d = 2\ell$. Here we will embed $\boldsymbol{S}$ in a larger matrix, compute the negative index for that larger matrix, and then relate that negative index to that for $\boldsymbol{S}$. First add another (block) row and column to $\boldsymbol{S}$ to obtain

$$\boldsymbol{S}_{\mathrm{new}} = \begin{pmatrix} \boldsymbol{S} & \boldsymbol{0} \\ \boldsymbol{0} & (-1)^{\ell}\boldsymbol{M}_{2\ell} \end{pmatrix},$$

and note that

$$\mathrm{n}(\boldsymbol{S}) = \mathrm{n}(\boldsymbol{S}_{\mathrm{new}}) - \mathrm{n}\big((-1)^{\ell}\boldsymbol{M}_{2\ell}\big) = \mathrm{n}(\boldsymbol{S}_{\mathrm{new}}) + \mathrm{n}\big((-1)^{\ell-1}\boldsymbol{M}_{2\ell}\big) - n. \tag{3.1}$$

A similar result holds for the positive index. Setting $\boldsymbol{A}$ to be the invertible block matrix

$$\boldsymbol{A} = \left(\begin{array}{c|c|c} \boldsymbol{I}_n & \boldsymbol{0} & \boldsymbol{0}_n \\ \hline \boldsymbol{0} & \boldsymbol{I}_{(2\ell-1)n} & \boldsymbol{0} \\ \hline \boldsymbol{0}_n & \boldsymbol{0}_{n\times(\ell-1)n} \ -\boldsymbol{I}_n \ \boldsymbol{0}_{n\times(\ell-1)n} & -\boldsymbol{I}_n \end{array}\right),$$

we clearly have $\mathrm{n}(\boldsymbol{S}_{\mathrm{new}}) = \mathrm{n}(\boldsymbol{A}\boldsymbol{S}_{\mathrm{new}}\boldsymbol{A}^{\mathrm{T}})$. It can be checked that

$$\boldsymbol{A}\boldsymbol{S}_{\mathrm{new}}\boldsymbol{A}^{\mathrm{T}} = \left(\begin{array}{c|c|c} \boldsymbol{M}_0 & \boldsymbol{0} & \boldsymbol{0} \\ \hline \boldsymbol{0} & \boldsymbol{S}_1 & \boldsymbol{S}_2 \\ \hline \boldsymbol{0} & \boldsymbol{S}_2^{\mathrm{H}} & \boldsymbol{0}_{\ell n} \end{array}\right),$$

where

$$\boldsymbol{S}_1 = \begin{pmatrix} \boldsymbol{M}_2 & \boldsymbol{M}_3 & \cdots & \boldsymbol{M}_{1+\ell} \\ -\boldsymbol{M}_3 & -\boldsymbol{M}_4 & \cdots & -\boldsymbol{M}_{2+\ell} \\ \vdots & \vdots & \vdots & \vdots \\ \pm\boldsymbol{M}_{1+\ell} & \pm\boldsymbol{M}_{2+\ell} & \cdots & \pm\boldsymbol{M}_{2\ell} \end{pmatrix},$$

and

$$\boldsymbol{S}_2 = \begin{pmatrix} \boldsymbol{M}_{2+\ell} & \boldsymbol{M}_{3+\ell} & \cdots & \boldsymbol{M}_{2\ell-1} & \boldsymbol{M}_{2\ell} & -\boldsymbol{M}_{1+\ell} \\ -\boldsymbol{M}_{3+\ell} & -\boldsymbol{M}_{4+\ell} & \cdots & \boldsymbol{M}_{2\ell} & \boldsymbol{0}_n & \boldsymbol{M}_{2+\ell} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \mp\boldsymbol{M}_{2\ell} & \boldsymbol{0}_n & \cdots & \boldsymbol{0}_n & \boldsymbol{0}_n & \pm\boldsymbol{M}_{2\ell-1} \\ \boldsymbol{0}_n & \boldsymbol{0}_n & \cdots & \boldsymbol{0}_n & \boldsymbol{0}_n & \mp\boldsymbol{M}_{2\ell} \end{pmatrix}.$$

The plus/minus is shorthand for $(-1)^{\ell-1}$.

The matrix $\boldsymbol{A}\boldsymbol{S}_{\text{new}}\boldsymbol{A}^{\mathrm{T}}$ has precisely the same block structure as the previously considered odd case, and the invertibility of $\boldsymbol{S}_2$ is guaranteed by the invertibility of $\boldsymbol{M}_{2\ell}$. We may then say that

$$\mathrm{n}\big(\boldsymbol{A}\boldsymbol{S}_{\text{new}}\boldsymbol{A}^{\mathrm{T}}\big)=\mathrm{n}(\boldsymbol{S}_{\text{new}})=\mathrm{n}(\boldsymbol{M}_0)+\ell n.$$

Combining this result with that of (3.1) yields the negative index for the original matrix $\boldsymbol{S}$,

$$\mathrm{n}(\boldsymbol{S})=\mathrm{n}(\boldsymbol{M}_0)+\mathrm{n}\big((-1)^{\ell-1}\boldsymbol{M}_{2\ell}\big)+(\ell-1)n.$$

Clearly, there is an analogous result for the positive index. □

The Hamiltonian–Krein index for the linearization is

$$K_{\text{Ham}}^{-}=\mathrm{n}(\boldsymbol{S}),\qquad K_{\text{Ham}}^{+}=\mathrm{p}(\boldsymbol{S}),$$

where $\mathrm{n}(\boldsymbol{S})$ and $\mathrm{p}(\boldsymbol{S})$ have been computed in Lemma 2.1. Before we can state the index theorem for the original polynomial, we need to relate the negative Krein index for the linearization (recall (2.2)) to that for the polynomial. Suppose that $\boldsymbol{x}_0$ is an eigenvector for the polynomial associated with the purely imaginary polynomial eigenvalue $\mathrm{i}\lambda_0$. The associated eigenvector for the linearization is $\boldsymbol{v}_0=(\boldsymbol{x}_0,\mathrm{i}\lambda_0\boldsymbol{x}_0,(\mathrm{i}\lambda_0)^2\boldsymbol{x}_0,\dots,(\mathrm{i}\lambda_0)^{n-1}\boldsymbol{x}_0)^{\mathrm{T}}$. It is not difficult to check that

$$\boldsymbol{S}\boldsymbol{v}_0=\begin{pmatrix}\boldsymbol{M}_0\boldsymbol{x}_0\\ +(\mathrm{i}\lambda_0\boldsymbol{M}_2+(\mathrm{i}\lambda_0)^2\boldsymbol{M}_3+\cdots+(\mathrm{i}\lambda_0)^{m-1}\boldsymbol{M}_d)\boldsymbol{x}_0\\ -(\mathrm{i}\lambda_0\boldsymbol{M}_3+(\mathrm{i}\lambda_0)^2\boldsymbol{M}_4+\cdots+(\mathrm{i}\lambda_0)^{m-2}\boldsymbol{M}_d)\boldsymbol{x}_0\\ \vdots\\ \mp\mathrm{i}\lambda_0\boldsymbol{M}_d\boldsymbol{x}_0\end{pmatrix},$$

so that

$$\begin{aligned}\boldsymbol{S}\boldsymbol{v}_0\cdot\boldsymbol{v}_0&=\boldsymbol{M}_0\boldsymbol{x}_0\cdot\boldsymbol{x}_0-\big[(\mathrm{i}\lambda_0)^2\boldsymbol{M}_2+(\mathrm{i}\lambda_0)^3\boldsymbol{M}_3+\cdots+(\mathrm{i}\lambda_0)^d\boldsymbol{M}_d\big]\boldsymbol{x}_0\cdot\boldsymbol{x}_0\\ &\quad-\big[(\mathrm{i}\lambda_0)^3\boldsymbol{M}_3+(\mathrm{i}\lambda_0)^4\boldsymbol{M}_4+\cdots+(\mathrm{i}\lambda_0)^d\boldsymbol{M}_d\big]\boldsymbol{x}_0\cdot\boldsymbol{x}_0-\cdots-(\mathrm{i}\lambda_0)^d\boldsymbol{M}_d\boldsymbol{x}_0\cdot\boldsymbol{x}_0\\ &=\big[\boldsymbol{M}_0-(\mathrm{i}\lambda_0)^2\boldsymbol{M}_2-2(\mathrm{i}\lambda_0)^3\boldsymbol{M}_3-\cdots-(m-1)(\mathrm{i}\lambda_0)^d\boldsymbol{M}_d\big]\boldsymbol{x}_0\cdot\boldsymbol{x}_0.\end{aligned}$$

Upon using the fact that

$$\boldsymbol{P}_d(\mathrm{i}\lambda_0)\boldsymbol{x}_0=\boldsymbol{0}\quad\Rightarrow\quad\boldsymbol{M}\boldsymbol{x}_0=-\big[\mathrm{i}\lambda_0\boldsymbol{M}_1+(\mathrm{i}\lambda_0)^2\boldsymbol{M}_2+\cdots+(\mathrm{i}\lambda_0)^d\boldsymbol{M}_d\big]\boldsymbol{x}_0,$$

and substituting the above expression for $\boldsymbol{M}\boldsymbol{x}_0$ into the expression for $\boldsymbol{S}\boldsymbol{v}_0\cdot\boldsymbol{v}_0$, we see that

$$\boldsymbol{S}\boldsymbol{v}_0\cdot\boldsymbol{v}_0=-\mathrm{i}\lambda_0\boldsymbol{P}_d'(\mathrm{i}\lambda_0)\boldsymbol{x}_0\cdot\boldsymbol{x}_0.$$

Thus, for a purely imaginary polynomial eigenvalue we will define the negative (positive) Krein index by

$$k_{\mathrm{i}}^{-}(\mathrm{i}\lambda_0)=\mathrm{n}\big(\big[-\mathrm{i}\lambda_0\boldsymbol{P}_d'(\mathrm{i}\lambda_0)\big]\big|_{\mathbb{E}_{\mathrm{i}\lambda_0}}\big),\qquad k_{\mathrm{i}}^{+}(\mathrm{i}\lambda_0)=\mathrm{p}\big(\big[-\mathrm{i}\lambda_0\boldsymbol{P}_d'(\mathrm{i}\lambda_0)\big]\big|_{\mathbb{E}_{\mathrm{i}\lambda_0}}\big),\tag{3.2}$$

where $\mathbb{E}_{\mathrm{i}\lambda_0}$ is the generalized eigenspace associated with the polynomial eigenvalue $\mathrm{i}\lambda_0$. Using (3.2), the total negative (positive) Krein index is given by

$$k_{\mathrm{i}}^{\pm}=\sum k_{\mathrm{i}}^{\pm}(\mathrm{i}\lambda_0).\tag{3.3}$$

Theorem 3.4. *Consider the ⋆-even polynomial (1.1), where $\boldsymbol{M}_0,\boldsymbol{M}_d$ are nonsingular. The Hamiltonian–Krein indices (2.3) for the polynomial satisfy*

$$K_{\text{Ham}}^{-}=\begin{cases}\mathrm{n}(\boldsymbol{M}_0)+(\ell-1)n, & d=2\ell-1,\\ \mathrm{n}(\boldsymbol{M}_0)+\mathrm{n}((-1)^{\ell-1}\boldsymbol{M}_{2\ell})+(\ell-1)n, & d=2\ell,\end{cases}$$

and

$$K_{\mathrm{Ham}}^{+} = \begin{cases} \mathrm{p}(\boldsymbol{M}_0) + (\ell-1)n, & d = 2\ell-1, \\ \mathrm{p}(\boldsymbol{M}_0) + \mathrm{p}((-1)^{\ell-1}\boldsymbol{M}_{2\ell}) + (\ell-1)n, & d = 2\ell. \end{cases}$$

Here the Krein index for a purely imaginary polynomial eigenvalue is defined in (3.2), and the total Krein index is given in (3.3).

Remark 3.5. In applications it may be the case that the matrix $\boldsymbol{M}_0$ is singular. In this case, when $d = 2$ it was shown in Bronski et al. [5] – which is (partially) concerned with the quadratic polynomial eigenvalue problem in a more general setting – that the negative Hamiltonian–Krein index satisfies

$$K_{\mathrm{Ham}}^{-} = \mathrm{n}(\boldsymbol{M}_0) + \mathrm{n}(\boldsymbol{M}_2) - \mathrm{n}\big((\boldsymbol{M}_2 - \boldsymbol{M}_1\boldsymbol{M}_0^{-1}\boldsymbol{M}_1)|_{\ker(\boldsymbol{M}_0)}\big).$$

The underlying assumptions leading to this result are that $\boldsymbol{M}_1 : \ker(\boldsymbol{M}_0) \mapsto \ker(\boldsymbol{M}_0)^{\perp}$, and the matrix $(\boldsymbol{M}_2 - \boldsymbol{M}_1\boldsymbol{M}_0^{-1}\boldsymbol{M}_1)|_{\ker(\boldsymbol{M}_0)}$ is invertible. While we will not do so herein, it is not difficult to generalize that result to the higher-order $\star$-even polynomials considered in this paper.

Remark 3.6. Since

$$k_{\mathrm{i}}^{-} - k_{\mathrm{i}}^{+} = K_{\mathrm{Ham}}^{-} - K_{\mathrm{Ham}}^{+},$$

we see from Theorem 3.4 that the purely imaginary polynomial eigenvalues satisfy

$$k_{\mathrm{i}}^{-} - k_{\mathrm{i}}^{+} = \begin{cases} \mathrm{n}(\boldsymbol{M}_0) - \mathrm{p}(\boldsymbol{M}_0), & d = 2\ell-1, \\ [\mathrm{n}(\boldsymbol{M}_0) - \mathrm{p}(\boldsymbol{M}_0)] + [\mathrm{n}((-1)^{\ell-1}\boldsymbol{M}_{2\ell}) - \mathrm{p}((-1)^{\ell-1}\boldsymbol{M}_{2\ell})], & d = 2\ell. \end{cases}$$

Remark 3.7. Chugunova and Pelinovsky [6] and Kollár [20] studied this polynomial eigenvalue problem for the quadratic case $d = 2$ under the additional assumption that $\boldsymbol{M}_2 > 0$. From Theorem 3.4 and Remark 3.6 we see

$$K_{\mathrm{Ham}}^{-} = \mathrm{n}(\boldsymbol{M}_0), \qquad K_{\mathrm{Ham}}^{+} = \mathrm{p}(\boldsymbol{M}_0) + n, \qquad k_{\mathrm{i}}^{+} = k_{\mathrm{i}}^{-} + n + \big[\mathrm{p}(\boldsymbol{M}_0) - \mathrm{n}(\boldsymbol{M}_0)\big].$$

The result for $K_{\mathrm{Ham}}^{\pm}$ is precisely that found in their works. However, their proofs are different than that presented herein.

It is interesting to note that the Hamiltonian–Krein index depends only on the lowest order term in the polynomial, the highest order term (if the degree of the polynomial is even), and the degree of the polynomial and size of the matrices if the degree is three or higher. The matrices $\boldsymbol{M}_1, \dots, \boldsymbol{M}_{d-1}$ influence the total number of polynomial eigenvalues with positive real part only as a second order effect.

Upon further reflection it is not surprising that the Hamiltonian–Krein indices should depend on n for $d \geqslant 3$. Consider the following illustrative example. Consider a polynomial of the form $\boldsymbol{M}_0 + \lambda^{2\ell}\boldsymbol{I}_n$, and suppose that $\boldsymbol{M}_0$ is positive definite. When $\ell = 1$, all of the polynomial eigenvalues will be purely imaginary, and $K_{\mathrm{Ham}}^{-} = 0$. If $\ell = 2$, then there will be precisely $2n$ polynomial eigenvalues with positive real part, which is in accordance with the theoretical result of $K_{\mathrm{Ham}}^{-} = 2n$. This result is clearly independent of the negative index of $\boldsymbol{M}_0$. Continuing in this fashion one more time, we see that when $\ell = 3$ there again will be $2n$ polynomial eigenvalues with positive real part, and $2n$ purely imaginary polynomial eigenvalues. From the theoretical result that $K_{\mathrm{Ham}}^{-} = 2n$, we know that all of the purely imaginary polynomial eigenvalues have positive Krein index.

3.2. Case study: Singularly perturbed $\star$-even polynomials

As a consequence of Lemma 2.1, for linear $\star$-even polynomials the number of eigenvalues with positive real part is bounded above by

$$k_{\mathrm{r}} + k_{\mathrm{c}} \leqslant \min\big\{\mathrm{n}(\boldsymbol{M}_0), \mathrm{p}(\boldsymbol{M}_0)\big\}. \tag{3.4}$$

Furthermore, as a consequence of Theorem 3.4, for quadratic $\star$-even polynomials the number of eigenvalues with positive real part is bounded above by

$$k_{\mathrm{r}} + k_{\mathrm{c}} \leqslant \min\{\mathrm{n}(\boldsymbol{M}_0) + \mathrm{n}(\boldsymbol{M}_2), \mathrm{p}(\boldsymbol{M}_0) + \mathrm{p}(\boldsymbol{M}_2)\}. \tag{3.5}$$

The result for the quadratic bound follows immediately from the equalities

$$k_{\mathrm{r}} + k_{\mathrm{c}} + k_{\mathrm{i}}^{-} = \mathrm{n}(\boldsymbol{M}_0) + \mathrm{n}(\boldsymbol{M}_2), \qquad k_{\mathrm{r}} + k_{\mathrm{c}} + k_{\mathrm{i}}^{+} = \mathrm{p}(\boldsymbol{M}_0) + \mathrm{p}(\boldsymbol{M}_2).$$

In particular, this upper bound is independent of the size of the matrices. This observation is the crucial one in the extension of the theory presented in this paper to the case where the matrix coefficients are actually either compact operators or operators with compact resolvent (see [5]). On the other hand, for higher-order polynomials the bound depends upon the size of the matrix coefficients. In this subsection we will show that the bound can be made independent of the matrix size for (some) singularly perturbed polynomials.

Let us consider a singulary perturbed cubic polynomial of the form

$$\boldsymbol{P}_3^{\epsilon}(\lambda) = \boldsymbol{M}_0 + \lambda \boldsymbol{M}_1 + \lambda^2 \boldsymbol{M}_2 + \epsilon\lambda^3 \boldsymbol{M}_3, \quad 0 < \epsilon \ll 1.$$

Assuming that $\boldsymbol{M}_0$ and $\boldsymbol{M}_3$ are nonsingular, we can invoke Theorem 3.4 to say that

$$K_{\mathrm{Ham}}^{-} = \mathrm{n}(\boldsymbol{M}_0) + n, \qquad K_{\mathrm{Ham}}^{+} = \mathrm{p}(\boldsymbol{M}_0) + n,$$

which yields the upper bound

$$k_{\mathrm{r}} + k_{\mathrm{c}} \leqslant \min\{\mathrm{n}(\boldsymbol{M}_0), \mathrm{p}(\boldsymbol{M}_0)\} + n.$$

We will proceed to refine this upper bound, and make it independent of the matrix size.

Using regular perturbation theory (see Kato [19]), there will be $2n$ polynomial eigenvalues which are $\mathcal{O}(1)$, and the other n polynomial eigenvalues will be $\mathcal{O}(1/\epsilon)$. The $\mathcal{O}(1)$ polynomial eigenvalues are to leading order found as solutions to the quadratic polynomial

$$\boldsymbol{P}_2^{0}(\lambda) = \boldsymbol{M}_0 + \lambda \boldsymbol{M}_1 + \lambda^2 \boldsymbol{M}_2,$$

and the other n polynomial eigenvalues are to leading order found by solving the linear polynomial

$$\boldsymbol{P}_1^{0}(z) = \boldsymbol{M}_2 + z\boldsymbol{M}_3, \qquad \lambda = z/\epsilon.$$

Under the additional assumption that $\boldsymbol{M}_2$ is nonsingular, we can use the results of (3.4) and (3.5) for each of these sub-polynomials to say that

$$\begin{aligned} \mathcal{O}(1)&: \quad k_{\mathrm{r}} + k_{\mathrm{c}} \leqslant \min\{\mathrm{n}(\boldsymbol{M}_0) + \mathrm{n}(\boldsymbol{M}_2), \mathrm{p}(\boldsymbol{M}_0) + \mathrm{p}(\boldsymbol{M}_2)\}, \\ \mathcal{O}(1/\epsilon)&: \quad k_{\mathrm{r}} + k_{\mathrm{c}} \leqslant \min\{\mathrm{n}(\boldsymbol{M}_2), \mathrm{p}(\boldsymbol{M}_2)\}. \end{aligned}$$

Adding these two quantities together, and noting that the polynomial eigenvalues for $\boldsymbol{P}_2^{0}(\lambda)$ and $\boldsymbol{P}_1^{0}(z)$ will not interact for $\epsilon > 0$ sufficiently small, gives us our first refined result:

Lemma 3.8. *Consider the singularly perturbed cubic polynomial $\boldsymbol{P}_3^{\epsilon}(\lambda)$ for $0 < \epsilon \ll 1$. Assume that the coefficients $\boldsymbol{M}_0$, $\boldsymbol{M}_2$, $\boldsymbol{M}_3$ are nonsingular. An upper bound for the number of polynomial eigenvalues with positive real part (independent of the matrix size) is given by*

$$k_{\mathrm{r}} + k_{\mathrm{c}} \leqslant \underbrace{\min\{\mathrm{n}(\boldsymbol{M}_0) + \mathrm{n}(\boldsymbol{M}_2), \mathrm{p}(\boldsymbol{M}_0) + \mathrm{p}(\boldsymbol{M}_2)\}}_{\mathcal{O}(1)} + \underbrace{\min\{\mathrm{n}(\boldsymbol{M}_2), \mathrm{p}(\boldsymbol{M}_2)\}}_{\mathcal{O}(1/\epsilon)}.$$

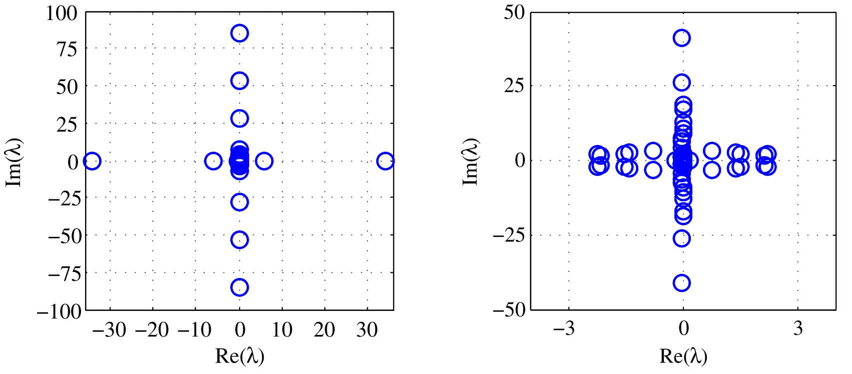

Fig. 2. (Color online.) A numerical demonstration of Lemma 3.8 when $n = 32$, and $\mathrm{n}(\boldsymbol{M}_0) = \mathrm{n}(\boldsymbol{M}_2) = 1$. The prediction is that for $0 < \epsilon \ll 1$ there will be at most three polynomial eigenvalues with positive real part. In the left panel, where $\epsilon = 0.01$, this upper bound is achieved with $k_\mathrm{r} = 3$ (the third positive polynomial eigenvalue is of $\mathcal{O}(10^{-1})$, and cannot be seen with the given axis scaling). In the right panel $\epsilon = 0.25$, and it is clear that ϵ is too large for Lemma 3.8 to be valid. In this case one must appeal to Theorem 3.4 to conclude that the upper bound is $k_\mathrm{r} + k_\mathrm{c} \leqslant 33$ (which is not achieved).

Remark 3.9. A further refinement is possible if it is assumed that $\boldsymbol{M}_2 > 0$, i.e., the quadratic polynomial $\boldsymbol{P}_2^0(\lambda)$ is (essentially) monic. In this case the $\mathcal{O}(1)$ polynomial eigenvalues have the upper bound

$$k_\mathrm{r} + k_\mathrm{c} \leqslant \mathrm{n}(\boldsymbol{M}_0).$$

Regarding the polynomial eigenvalues for $\boldsymbol{P}_1^0(z)$, by Theorem 3.4 the Hamiltonian–Krein index satisfies $K_{\mathrm{Ham}}^- = 0$, so that they will all be purely imaginary and have the same Krein index. As a consequence of the spectral symmetry $\{\lambda, -\bar{\lambda}\}$ for the polynomial eigenvalues, and the fact that the Krein index of a purely imaginary polynomial eigenvalue is an invariant under perturbation, for small $\epsilon > 0$ they will remain purely imaginary for the full polynomial $\boldsymbol{P}_3^\epsilon(\lambda)$. In conclusion, under the additional assumption of $\boldsymbol{M}_2 > 0$ we know that for $\boldsymbol{P}_3^\epsilon(\lambda)$ the unstable polynomial eigenvalues have the upper bound

$$k_\mathrm{r} + k_\mathrm{c} \leqslant \mathrm{n}(\boldsymbol{M}_0),$$

are of $\mathcal{O}(1)$, and are realized as perturbations of the polynomial eigenvalues for the reduced polynomial $\boldsymbol{P}_2^0(\lambda)$.

A numerical demonstration of the applicability of Lemma 3.8 is given in Fig. 2. For this figure the polynomial eigenvalues when $n = 32$ and $\mathrm{n}(\boldsymbol{M}_0) = \mathrm{n}(\boldsymbol{M}_2) = 1$ are calculated using the MATLAB command "polyeig". In the left panel $\epsilon = 0.01$, and in the right panel $\epsilon = 0.25$. The result of Lemma 3.8 predicts that for $\epsilon > 0$ sufficiently small there will be at most two polynomial eigenvalues of $\mathcal{O}(1)$ with positive real part, and at most one polynomial eigenvalue of $\mathcal{O}(1/\epsilon)$ with positive real part. This upper bound is achieved when $\epsilon = 0.01$ (the other positive real polynomial eigenvalue is of $\mathcal{O}(10^{-1})$, and cannot be seen with the given axis scaling). On the other hand, the result is clearly no longer applicable in the right panel, and instead we must appeal directly to Theorem 3.4, which gives an upper bound of 33 unstable polynomial eigenvalues (this upper bound is not achieved).

The additional unstable polynomial eigenvalues for larger ϵ arise from a series of Hamiltonian–Hopf bifurcations, i.e., the collision of purely imaginary polynomial eigenvalues of opposite Krein index. For $0 < \epsilon \ll 1$ the purely imaginary polynomial eigenvalues of $\mathcal{O}(1/\epsilon)$ have negative index, while those of $\mathcal{O}(1)$ have positive index. As ϵ increases these eigenvalues move towards each other, and eventually collide, which causes the bifurcation.

In order to see the separation of the purely imaginary polynomial eigenvalues based upon their Krein index, consider the following argument. The $\mathcal{O}(1)$ polynomial eigenvalues are to leading order realized as polynomial eigenvalues of the quadratic polynomial $\boldsymbol{P}_2^0(\lambda)$. By Theorem 3.4 the Hamiltonian–Krein index for this polynomial is

$$k_\mathrm{r} + k_\mathrm{c} + k_\mathrm{i}^- = \mathrm{n}(\boldsymbol{M}_0) + \mathrm{n}(\boldsymbol{M}_2),$$

and by Remark 3.6 we have

$$k_\mathrm{i}^+ = k_\mathrm{i}^- + \left[\mathrm{p}(\boldsymbol{M}_0) - \mathrm{n}(\boldsymbol{M}_0)\right] + \left[\mathrm{p}(\boldsymbol{M}_2) - \mathrm{n}(\boldsymbol{M}_2)\right].$$

For Fig. 2, where $\mathrm{n}(\boldsymbol{M}_0) = \mathrm{n}(\boldsymbol{M}_2) = 1$ with $n = 32$, these equalities become

$$k_\mathrm{r} + k_\mathrm{c} + k_\mathrm{i}^- = 2, \qquad k_\mathrm{i}^+ = k_\mathrm{i}^- + 60.$$

Clearly, the vast majority of the purely imaginary polynomial eigenvalues have positive Krein index. Since the index is an invariant under perturbation, the number k_i^+ can reduce if and only if an eigenvalue with positive index collides with an eigenvalue of negative index.

Now consider the $\mathcal{O}(1/\epsilon)$ polynomial eigenvalues, which to leading order are realized as polynomial eigenvalues of the linear polynomial $\boldsymbol{P}_1^0(z)$. By Theorem 3.4 the Hamiltonian–Krein index for this polynomial is

$$k_\mathrm{r} + k_\mathrm{c} + k_\mathrm{i}^- = \mathrm{n}(\boldsymbol{M}_2),$$

and by Remark 3.6 we have

$$k_\mathrm{i}^+ = k_\mathrm{i}^- + \left[\mathrm{p}(\boldsymbol{M}_0) - \mathrm{n}(\boldsymbol{M}_0)\right].$$

For Fig. 2, where $\mathrm{n}(\boldsymbol{M}_0) = \mathrm{n}(\boldsymbol{M}_2) = 1$ with $n = 32$, these equalities become

$$k_\mathrm{r} + k_\mathrm{c} + k_\mathrm{i}^- = 1, \qquad k_\mathrm{i}^+ = k_\mathrm{i}^- + 30.$$

Since k_c, $k_\mathrm{i}^\pm$ are even integers, we see for the numerical example that $k_\mathrm{r} = 1$ with $k_\mathrm{i}^+ = 30$. Thus, it now appears to be the case that all of the purely imaginary polynomial eigenvalues of $\mathcal{O}(1/\epsilon)$ have positive index. We now demonstrate that appearances are deceiving, and that instead they all have negative index.

Going back to (3.2), the Krein index for a simple polynomial eigenvalue is given by

$$k_\mathrm{i}^-(\mathrm{i}\lambda_0) = \mathrm{n}\big(-\mathrm{i}\lambda_0 \boldsymbol{P}_d'(\mathrm{i}\lambda_0)\boldsymbol{x}_0 \cdot \boldsymbol{x}_0\big), \qquad k_\mathrm{i}^+(\mathrm{i}\lambda_0) = \mathrm{p}\big(-\mathrm{i}\lambda_0 \boldsymbol{P}_d'(\mathrm{i}\lambda_0)\boldsymbol{x}_0 \cdot \boldsymbol{x}_0\big),$$

where $\mathrm{i}\lambda_0$ is the polynomial eigenvalue with associated eigenvector $\boldsymbol{x}_0$. For the linear polynomial we have

$$-\mathrm{i}z_0\big(\boldsymbol{P}_1^0\big)'(\mathrm{i}z_0)\boldsymbol{x}_0 \cdot \boldsymbol{x}_0 = -\mathrm{i}z_0 \boldsymbol{M}_3 \boldsymbol{x}_0 \cdot \boldsymbol{x}_0.$$

On the other hand, after using $z = \lambda/\epsilon$ to rewrite the cubic polynomial as

$$\boldsymbol{P}_3^\epsilon(z) = \epsilon^2 \boldsymbol{M}_0 + \epsilon z \boldsymbol{M}_1 + z^2 \boldsymbol{M}_2 + z^3 \boldsymbol{M}_3,$$

we see

$$-\mathrm{i}z_0\big(\boldsymbol{P}_3^\epsilon\big)'(\mathrm{i}z_0)\boldsymbol{x}_0 \cdot \boldsymbol{x}_0 = -\mathrm{i}z_0\big(\epsilon \boldsymbol{M}_1 + 2\mathrm{i}z_0 \boldsymbol{M}_2 - 3z_0^2 \boldsymbol{M}_3\big)\boldsymbol{x}_0 \cdot \boldsymbol{x}_0 \sim (\mathrm{i}z_0)z_0^2 \boldsymbol{M}_3 \boldsymbol{x}_0 \cdot \boldsymbol{x}_0.$$

The last equality arises from the fact that $\boldsymbol{M}_2\boldsymbol{x}_0 = -\mathrm{i}z_0\boldsymbol{M}_3\boldsymbol{x}_0$. Since $z_0^2 > 0$ we can conclude that

$$\mathrm{sign}\big(-\mathrm{i}z_0\big(\boldsymbol{P}_3^\epsilon\big)'(\mathrm{i}z_0)\boldsymbol{x}_0 \cdot \boldsymbol{x}_0\big) = -\,\mathrm{sign}\big(-\mathrm{i}z_0\big(\boldsymbol{P}_1^0\big)'(\mathrm{i}z_0)\boldsymbol{x}_0 \cdot \boldsymbol{x}_0\big);$$

in other words, polynomial eigenvalues with positive Krein index for the reduced polynomial $\boldsymbol{P}_1^0(z)$ correspond to polynomial eigenvalues with negative Krein index for the full polynomial $\boldsymbol{P}_3^\epsilon(z)$. Thus, for the numerical example we have 30 purely imaginary eigenvalues of $\mathcal{O}(1/\epsilon)$ with negative Krein index.

4. Indices for Hermitian polynomials

In this section we assume that all of the matrix coefficients are Hermitian. We wish to derive polynomial eigenvalue indices for these polynomials. The nature of the index will depend on assumptions associated with the inner coefficients $\boldsymbol{M}_1, \ldots, \boldsymbol{M}_{d-1}$.

4.1. Positive real-valued polynomial eigenvalues

We begin by noting that in the proof of Lemma 3.1 we uncovered a general rule concerning the number of positive real-valued polynomial eigenvalues for Hermitian polynomials.

Lemma 4.1. *Consider the Hermitian polynomial*

$$\boldsymbol{P}_d(\lambda) = \sum_{j=0}^{d} \lambda^j \boldsymbol{M}_j,$$

where each $\boldsymbol{M}_j \in \mathbb{C}^{n\times n}$ is Hermitian. Suppose that $\boldsymbol{M}_0$ and $\boldsymbol{M}_d$ are nonsingular. Then

$$k_{\mathrm{r}} \geqslant \left|\mathrm{n}(\boldsymbol{M}_d) - \mathrm{n}(\boldsymbol{M}_0)\right|.$$

Proof. Consider $\boldsymbol{P}_d(\lambda)$ for $\lambda \in \mathbb{R}$. Since the polynomial is Hermitian, all of the eigenvalues for $\boldsymbol{P}_d(\lambda)$, denoted by $r_j(\lambda)$ for $j = 1, \ldots, n$, must be real-valued. Since the polynomial itself is Hermitian for real-valued λ, each eigenvalue is real analytic. For large λ we have

$$\frac{\boldsymbol{P}_d(\lambda)}{\lambda^d} = \boldsymbol{M}_d + \mathcal{O}\left(\lambda^{-1}\right);$$

hence, for large positive λ, $\mathrm{n}(\boldsymbol{M}_d)$ of the eigenvalues are negative and $\mathrm{p}(\boldsymbol{M}_d) = n - \mathrm{n}(\boldsymbol{M}_d)$ are positive. Since $\boldsymbol{P}_d(0) = \boldsymbol{M}_0$, it must further be the case that $r_j(0) \in \sigma(\boldsymbol{M}_0)$ for each j. Thus, $\mathrm{n}(\boldsymbol{M}_0)$ of the eigenvalues will be negative at $\lambda = 0$, and the other $\mathrm{p}(\boldsymbol{M}_0)$ will be positive. We now see by continuity that there will be at least $|\mathrm{n}(\boldsymbol{M}_d) - \mathrm{n}(\boldsymbol{M}_0)|$ solutions to the system of equations $r_j(\lambda) = 0$, $j = 1, \ldots, n$, for real $\lambda > 0$. Since $r_j(\lambda) = 0$ if and only if $\det[\boldsymbol{P}_d(\lambda)] = 0$, the result now follows. □

The same proof also allows us to restate the result as

$$k_{\mathrm{r}} \geqslant \left|\mathrm{p}(\boldsymbol{M}_d) - \mathrm{p}(\boldsymbol{M}_0)\right|.$$

Furthermore, another proof of Lemma 4.1 is available by considering the Hermitian linearization of the polynomial given by Mehrmann and Watkins [31, p. 113], and then applying the results for linear polynomials given by Grillakis [11].

4.2. General result

Ideally, in addition to the lower bound on the number of real-valued positive polynomial eigenvalues given in Lemma 4.1, we would like to determine the total number of polynomial eigenvalues with positive real part. Our initial result will not allow us to provide that information; however, it will allow us to say something about the distribution of the polynomial eigenvalues. We first set $\lambda = \mathrm{i}\gamma$, so that the Hermitian polynomial becomes the $\star$-even polynomial

$$\boldsymbol{P}_d^{\star}(\gamma) = \sum_{j=0}^{d} \mathrm{i}^j \gamma^j \boldsymbol{M}_j = \boldsymbol{M}_0 + \gamma(\mathrm{i}\boldsymbol{M}_1) - \gamma^2 \boldsymbol{M}_2 - \gamma^3(\mathrm{i}\boldsymbol{M}_3) + \gamma^4 \boldsymbol{M}_4 + \cdots.$$

For this $\star$-even polynomial the polynomial eigenvalues satisfy the $\{\gamma, -\overline{\gamma}\}$ symmetry, which for the Hermitian polynomial yields the $\{\lambda, \bar{\lambda}\}$ symmetry. In general, polynomial eigenvalues in the closed right-half of the γ-plane for the $\star$-even polynomial correspond to polynomial eigenvalues in the closed lower-half of the λ-plane.

Regarding the Hamiltonian–Krein index for the $\star$-even polynomial $\boldsymbol{P}_d^*(\gamma)$, we can invoke Theorem 3.4 to explicitly say

$$K_{\mathrm{Ham}}^{\pm}(\gamma) = k_{\mathrm{r}}(\gamma) + k_{\mathrm{c}}(\gamma) + k_{\mathrm{i}}^{\pm}(\gamma).$$

The count for the Hermitian polynomial then requires us to simply respect the spectral transformation. First,

$$\kappa_{\mathrm{i}}^{-}(\lambda) = k_{\mathrm{r}}(\gamma),$$

where $\kappa_{\mathrm{i}}^{-}(\lambda)$ refers to the number of purely imaginary polynomial eigenvalues for the Hermitian polynomial in the lower-half plane. Regarding the complex-valued polynomial eigenvalues with positive real part for the $\star$-even polynomial, they will become complex-valued polynomial eigenvalues with negative imaginary part and nonzero real part. Unfortunately, with this argument we cannot know how many of these polynomial eigenvalues will also have positive real part. In conclusion,

$$\kappa_{\mathrm{c}}^{-}(\lambda) = k_{\mathrm{c}}(\gamma),$$

where $\kappa_{\mathrm{c}}^{-}(\lambda)$ counts the number of polynomial eigenvalues for the Hermitian polynomial with nonzero real part and negative imaginary part.

Finally, consider the fate of the purely imaginary polynomial eigenvalues for the $\star$-even polynomial. These will map to purely real polynomial eigenvalues for the Hermitian polynomial. Furthermore, since

$$-\gamma \boldsymbol{P}_d^{\star}(\gamma) = -\lambda \boldsymbol{P}_d(\lambda), \quad \gamma \in \mathrm{i}\mathbb{R}\ (\lambda \in \mathbb{R}), \tag{4.1}$$

for the Hermitian polynomial we now talk about the Krein index for purely real polynomial eigenvalues. Analogously to (3.2)–(3.3), we define the negative (positive) Krein index for a purely real polynomial eigenvalue λ_0 as

$$\kappa_{\mathrm{r}}^{-}(\lambda_0) = \mathrm{n}\big(\big[-\lambda_0 \boldsymbol{P}_d'(\lambda_0)\big]\big|_{\mathbb{E}_{\lambda_0}}\big), \qquad \kappa_{\mathrm{r}}^{+}(\lambda_0) = \mathrm{p}\big(\big[-\lambda_0 \boldsymbol{P}_d'(\lambda_0)\big]\big|_{\mathbb{E}_{\lambda_0}}\big), \tag{4.2}$$

where $\mathbb{E}_{\lambda_0}$ is the generalized eigenspace associated with the polynomial eigenvalue λ_0. The total negative (positive) Krein index is

$$\kappa_{\mathrm{r}}^{\pm} = \sum \kappa_{\mathrm{r}}^{\pm}(\lambda_0). \tag{4.3}$$

As a consequence of (4.1) we can now say

$$\kappa_{\mathrm{r}}^{\pm}(\lambda) = k_{\mathrm{i}}^{\pm}(\gamma),$$

where here $\kappa_{\mathrm{r}}^{\pm}(\lambda)$ is understood to be the total negative (positive) Krein index for the Hermitian polynomial.

From the above discussion, and applying Theorem 3.4 upon noting that $(\mathrm{i})^{2\ell} = (-1)^{\ell}$ implies

$$\mathrm{n}\big((-1)^{\ell-1}(\mathrm{i})^{2\ell}\boldsymbol{M}_{2\ell}\big) = \mathrm{p}(\boldsymbol{M}_{2\ell}), \qquad \mathrm{p}\big((-1)^{\ell-1}(\mathrm{i})^{2\ell}\boldsymbol{M}_{2\ell}\big) = \mathrm{n}(\boldsymbol{M}_{2\ell}),$$

we can now state our index theorem for the Hermitian polynomial:

Theorem 4.2. *Consider the Hermitian polynomial of Lemma 4.1, where $\boldsymbol{M}_0$, $\boldsymbol{M}_d$ are nonsingular. The spectral indices (2.3) for the polynomial satisfy*

$$\kappa_{\mathrm{r}}^{-} + \kappa_{\mathrm{c}}^{-} + \kappa_{\mathrm{i}}^{-} = \begin{cases} \mathrm{n}(\boldsymbol{M}_0) + (\ell-1)n, & d = 2\ell-1, \\ \mathrm{n}(\boldsymbol{M}_0) + \mathrm{p}(\boldsymbol{M}_{2\ell}) + (\ell-1)n, & d = 2\ell, \end{cases}$$

and

$$\kappa_{\mathrm{r}}^{+} + \kappa_{\mathrm{c}}^{-} + \kappa_{\mathrm{i}}^{-} = \begin{cases} \mathrm{p}(\boldsymbol{M}_0) + (\ell-1)n, & d = 2\ell-1, \\ \mathrm{p}(\boldsymbol{M}_0) + \mathrm{n}(\boldsymbol{M}_{2\ell}) + (\ell-1)n, & d = 2\ell. \end{cases}$$

Here κ_{c}^{-} is the total number of complex-valued polynomial eigenvalues with negative imaginary part, κ_{i}^{-} is the total number of purely imaginary polynomial eigenvalues with negative imaginary part, and $\kappa_{\mathrm{r}}^{\pm}$ are the total number of real polynomial eigenvalues with positive (negative) Krein index (see (4.2) and (4.3)).

Remark 4.3. The analogue of Remark 3.6 is

$$\kappa_{\mathrm{r}}^{-} - \kappa_{\mathrm{r}}^{+} = \begin{cases} \mathrm{n}(\boldsymbol{M}_0) - \mathrm{p}(\boldsymbol{M}_0), & d = 2\ell-1, \\ 2[\mathrm{n}(\boldsymbol{M}_0) - \mathrm{n}(\boldsymbol{M}_{2\ell})], & d = 2\ell. \end{cases}$$

The equality for d even follows from

$$\left[\mathrm{n}(\boldsymbol{M}_0) - \mathrm{p}(\boldsymbol{M}_0)\right] + \left[\mathrm{p}(\boldsymbol{M}_{2\ell}) - \mathrm{n}(\boldsymbol{M}_{2\ell})\right] = 2\left[\mathrm{n}(\boldsymbol{M}_0) - \mathrm{n}(\boldsymbol{M}_{2\ell})\right].$$

Since the total number of real-valued polynomial eigenvalues is given by $\kappa_{\mathrm{r}} = \kappa_{\mathrm{r}}^{-} + \kappa_{\mathrm{r}}^{+}$, we have

$$\kappa_{\mathrm{r}} = 2\kappa_{\mathrm{r}}^{+} + \begin{cases} \mathrm{n}(\boldsymbol{M}_0) - \mathrm{p}(\boldsymbol{M}_0), & d = 2\ell-1, \\ 2[\mathrm{n}(\boldsymbol{M}_0) - \mathrm{n}(\boldsymbol{M}_{2\ell})], & d = 2\ell. \end{cases}$$

In particular, Hermitian polynomials of even degree necessarily have an even number of real-valued polynomial eigenvalues. This is a generalization of the well-known result that even-order polynomials with real-valued coefficients have an even number of zeros (counting multiplicity).

Remark 4.4. Recently Kollár and Miller [21] considered the spectral problem for Hermitian polynomials via a graphical approach. They showed that the Krein index of a purely real polynomial eigenvalue can be determined by carefully studying the graphs of the eigenvalues $r_j(\lambda)$ given in the proof of Lemma 4.1. Via this graphical approach they are also able to prove the instability result Lemma 4.1, as well as analogues to Theorem 4.2 and Remark 4.3 when $d = 2\ell - 1$.

4.3. Case study: Hermitian polynomials as a perturbation of $\star$-even polynomials

Unfortunately, Theorem 4.2 does not provide an instability result, in that it provides no information as to how many polynomial eigenvalues have positive real part. We will now attack the problem from a different perspective. The final results will not necessarily be as robust as that of Theorem 4.2, in that we will impose constraints on the coefficients $\boldsymbol{M}_1, \dots, \boldsymbol{M}_{d-1}$. We will derive our results via a perturbative approach; namely, we will consider the polynomial to be a perturbation of a $\star$-even matrix polynomial, and then derive conditions which ensure:

(a) there are no purely imaginary polynomial eigenvalues (i.e., $\kappa_{\mathrm{i}}^{-} = 0$ in Theorem 4.2);
(b) all of the purely imaginary polynomial eigenvalues with negative (positive) Krein index for the unperturbed $\star$-even problem move into the right-half of the complex plane upon applying the perturbation.

In the case of a quadratic Hermitian polynomial, these conditions are ensured if the (damping) matrix $\boldsymbol{M}_1$ is assumed to be definite, which is not an unusual assumption. We consider problems in Section 4.4 for which these assumptions are dropped, but instead we make assumptions concerning the relative size of the polynomial coefficients.

First assume that $d = 2\ell$. We begin by considering the polynomial as being embedded in the family of Hermitian polynomials

$$\boldsymbol{P}^{\epsilon}(\lambda) = \boldsymbol{P}_{\mathrm{even}}(\lambda) + \epsilon \boldsymbol{P}_{\mathrm{odd}}(\lambda),$$

where

$$\boldsymbol{P}_{\text{even}}(\lambda)=\sum_{j=0}^{\ell}\lambda^{2j}\boldsymbol{M}_{2j},\qquad \boldsymbol{P}_{\text{odd}}(\lambda)=\sum_{j=0}^{\ell-1}\lambda^{2j+1}\boldsymbol{M}_{2j+1},$$

and $\epsilon\geqslant 0$ is a parameter. Note that $\boldsymbol{P}^1(\lambda)=\boldsymbol{P}_{2\ell}(\lambda)$. The idea of the proof is as follows. When $\epsilon=0$ the polynomial $\boldsymbol{P}^0(\lambda)$ is a $\star$-even polynomial, and consequently we can precisely determine the number of polynomial eigenvalues with positive real part via the Hamiltonian–Krein index. If for $\epsilon>0$ the perturbation $\boldsymbol{P}_{\text{odd}}(\lambda)$ ensures that (a) and (b) above hold, it will then be the case that the Hamiltonian–Krein index is unchanged for all $\epsilon>0$. Moreover, when $\epsilon>0$ the Hamiltonian–Krein index will count only those polynomial eigenvalues with positive real part.

We first consider (a). Suppose that $\mathrm{i}\lambda_0\in\mathrm{i}\mathbb{R}$ be a polynomial eigenvalue with associated eigenvector $\boldsymbol{v}_0$. We have

$$0=\boldsymbol{P}_\epsilon(\mathrm{i}\lambda_0)\boldsymbol{v}_0\cdot\boldsymbol{v}_0=\sum_{j=0}^{\ell}(-1)^j\lambda_0^{2j}\boldsymbol{M}_{2j}\boldsymbol{v}_0\cdot\boldsymbol{v}_0+\mathrm{i}\epsilon\lambda_0\sum_{j=0}^{\ell-1}(-1)^j\lambda_0^{2j}\boldsymbol{M}_{2j+1}\boldsymbol{v}_0\cdot\boldsymbol{v}_0.$$

Since all of the matrices are Hermitian, the first term in the sum is purely real, and the second term is purely imaginary. Consequently, if we assume that for all $\mu\in\mathbb{R}$ and $\boldsymbol{v}\in\mathbb{C}^n$,

$$\sum_{j=0}^{\ell-1}(-1)^j\mu^{2j}\boldsymbol{M}_{2j+1}\boldsymbol{v}\cdot\boldsymbol{v}=\frac{\boldsymbol{P}_{\text{odd}}(\mathrm{i}\mu)}{\mathrm{i}\mu}\boldsymbol{v}\cdot\boldsymbol{v}\neq 0,\tag{4.4}$$

then there can be no purely imaginary polynomial eigenvalues for any $\epsilon>0$. Note that (4.4) is ensured, e.g., if we assume that the matrices alternate in their definiteness (for example, $\boldsymbol{M}_1$ is positive definite, $\boldsymbol{M}_3$ is negative definite, $\boldsymbol{M}_5$ is positive definite, etc.).

Now we consider point (b). Let $\mathrm{i}\lambda_0\in\mathrm{i}\mathbb{R}$ be an algebraically simple polynomial eigenvalue for the polynomial when $\epsilon=0$, and let the associated eigenspace be denoted by $\mathbb{E}_{\mathrm{i}\lambda_0}$. The perturbed polynomial eigenvalue is given by the expansion $\lambda=\mathrm{i}\lambda_0+\epsilon\lambda_1+\mathcal{O}(\epsilon^2)$, and the perturbed eigenvector is given by $\boldsymbol{v}=\boldsymbol{v}_0+\epsilon\boldsymbol{v}_1+\mathcal{O}(\epsilon^2)$, where $\boldsymbol{v}_0\in\mathbb{E}_{\mathrm{i}\lambda_0}$. Plugging this expansion into the polynomial yields that at $\mathcal{O}(\epsilon)$,

$$\boldsymbol{P}_{\text{even}}(\mathrm{i}\lambda_0)\boldsymbol{v}_1=-\lambda_1\boldsymbol{P}'_{\text{even}}(\mathrm{i}\lambda_0)\boldsymbol{v}_0-\boldsymbol{P}_{\text{odd}}(\mathrm{i}\lambda_0)\boldsymbol{v}_0.$$

Since $\boldsymbol{P}_{\text{even}}(\mathrm{i}\lambda_0)$ is a Hermitian matrix, the standard Fredholm solvability condition yields

$$\det\bigl(\lambda_1\bigl[-\mathrm{i}\lambda_0\boldsymbol{P}'_{\text{even}}(\mathrm{i}\lambda_0)\bigr]\big|_{\mathbb{E}_{\mathrm{i}\lambda_0}}+\bigl[-\mathrm{i}\lambda_0\boldsymbol{P}_{\text{odd}}(\mathrm{i}\lambda_0)\bigr]\big|_{\mathbb{E}_{\mathrm{i}\lambda_0}}\bigr)=0.\tag{4.5}$$

Regarding the matrix $-\mathrm{i}\lambda_0\boldsymbol{P}'_{\text{even}}(\mathrm{i}\lambda_0)|_{\mathbb{E}_{\mathrm{i}\lambda_0}}$, this is what is used when computing the Krein indices of $\mathrm{i}\lambda_0$ (see (3.2)),

$$k_{\mathrm{i}}^-(\lambda_0)=\mathrm{n}\bigl(-\mathrm{i}\lambda_0\boldsymbol{P}'_{\text{even}}(\mathrm{i}\lambda_0)\big|_{\mathbb{E}_{\mathrm{i}\lambda_0}}\bigr),\qquad k_{\mathrm{i}}^+(\lambda_0)=\mathrm{p}\bigl(-\mathrm{i}\lambda_0\boldsymbol{P}'_{\text{even}}(\mathrm{i}\lambda_0)\big|_{\mathbb{E}_{\mathrm{i}\lambda_0}}\bigr).$$

If we assume that the other matrix $-\mathrm{i}\lambda_0\boldsymbol{P}_{\text{odd}}(\mathrm{i}\lambda_0)|_{\mathbb{E}_{\mathrm{i}\lambda_0}}$ is positive definite, then it will be the case that $k_{\mathrm{i}}^-(\mathrm{i}\lambda_0)$ of the solutions to (4.5) will be positive, and the others will be negative. On the other hand, if the other matrix is negative definite, then $k_{\mathrm{i}}^+(\mathrm{i}\lambda_0)$ of the solutions to (4.5) will be positive, and the others will be negative. Since

$$-\mathrm{i}\lambda_0\boldsymbol{P}_{\text{odd}}(\mathrm{i}\lambda_0)=\lambda_0^2\sum_{j=0}^{\ell-1}(-1)^j\lambda_0^{2j}\boldsymbol{M}_{2j+1},$$

the positive (negative) definiteness will be guaranteed if $(-1)^j\boldsymbol{M}_{2j+1}$ is positive (negative) definite for $j=0,\dots,\ell-1$. Of course, this condition also ensures that (4.4) holds.

Assuming definiteness, summing up over all purely imaginary polynomial eigenvalues reveals that for small $\epsilon > 0$ precisely $k_{\mathrm{i}}^{\pm}$ of these polynomial eigenvalues will move into the right-half plane. Since they cannot cross the imaginary axis for any $\epsilon > 0$, they will remain in the right-half plane for all $\epsilon > 0$. Regarding those polynomial eigenvalues which are in the right-half plane when $\epsilon = 0$, since they cannot cross the imaginary axis they must remain in the right-half plane for all $\epsilon > 0$. Finally, any polynomial eigenvalues which originally have negative real part must continue to have this property for all ϵ. In conclusion, there will be precisely $K_{\mathrm{Ham}}^{\pm}$ polynomial eigenvalues for the polynomial $\boldsymbol{P}^{\epsilon}(\lambda)$ with positive real part for any $\epsilon > 0$, which gives us the following result.

Lemma 4.5. *Consider the Hermitian polynomial of Lemma 4.1 with $d = 2\ell$. Suppose that all of the purely imaginary polynomial eigenvalues for the ($\star$-even) polynomial $\boldsymbol{P}_{\mathrm{even}}(\lambda)$ are algebraically simple. If for all $\mu \in \mathbb{R}$, $-\mathrm{i}\mu\boldsymbol{P}_{\mathrm{odd}}(\mathrm{i}\mu) > 0$, then the instability index for the Hermitian matrix polynomial is*

$$k_{\mathrm{r}} + k_{\mathrm{c}} = \mathrm{n}(\boldsymbol{M}_0) + \mathrm{n}\big((-1)^{\ell-1}\boldsymbol{M}_{2\ell}\big) + (\ell - 1)n,$$

while if $-\mathrm{i}\mu\boldsymbol{P}_{\mathrm{odd}}(\mathrm{i}\mu) < 0$, the instability index is

$$k_{\mathrm{r}} + k_{\mathrm{c}} = \mathrm{p}(\boldsymbol{M}_0) + \mathrm{p}\big((-1)^{\ell-1}\boldsymbol{M}_{2\ell}\big) + (\ell - 1)n.$$

As a useful corollary for quadratic Hermitian polynomials, we have the following result which combines the result of Lemma 4.1 with that of Lemma 4.5:

Corollary 4.6. *Consider the quadratic Hermitian polynomial*

$$\boldsymbol{P}_2(\lambda) = \boldsymbol{M}_0 + \lambda\boldsymbol{M}_1 + \lambda^2\boldsymbol{M}_2.$$

The total number of positive and real-valued positive polynomial eigenvalues has the lower bound

$$k_{\mathrm{r}} \geqslant \big|\mathrm{n}(\boldsymbol{M}_2) - \mathrm{n}(\boldsymbol{M}_0)\big| \quad \big(= \big|\mathrm{p}(\boldsymbol{M}_2) - \mathrm{p}(\boldsymbol{M}_0)\big|\big).$$

If $\boldsymbol{M}_1$ is definite, and if all of the purely imaginary polynomial eigenvalues of $\boldsymbol{M}_0 + \lambda^2\boldsymbol{M}_2$ are algebraically simple, the total number of unstable eigenvalues satisfy the count

$$\boldsymbol{M}_1 > 0: \quad k_{\mathrm{r}} + k_{\mathrm{c}} = \mathrm{n}(\boldsymbol{M}_0) + \mathrm{n}(\boldsymbol{M}_2); \qquad \boldsymbol{M}_1 < 0: \quad k_{\mathrm{r}} + k_{\mathrm{c}} = \mathrm{p}(\boldsymbol{M}_0) + \mathrm{p}(\boldsymbol{M}_2).$$

In particular, if $\boldsymbol{M}_2$ has the same definiteness properties as $\boldsymbol{M}_1$, then all of the polynomial eigenvalues are real-valued and satisfy the index

$$\boldsymbol{M}_1, \boldsymbol{M}_2 > 0: \quad k_{\mathrm{r}} = \mathrm{n}(\boldsymbol{M}_0); \qquad \boldsymbol{M}_1, \boldsymbol{M}_2 < 0: \quad k_{\mathrm{r}} = \mathrm{p}(\boldsymbol{M}_0).$$

Now suppose that $d = 2\ell + 1$ is odd, and set

$$\boldsymbol{P}_{\mathrm{even}}(\lambda) = \sum_{j=0}^{\ell} \lambda^{2j}\boldsymbol{M}_{2j}, \qquad \boldsymbol{P}_{\mathrm{odd}}(\lambda) = \sum_{j=0}^{\ell} \lambda^{2j+1}\boldsymbol{M}_{2j+1}.$$

Since all polynomial eigenvalues must be nonzero, we embed the original polynomial in the family of rational polynomials,

$$\boldsymbol{R}^{\epsilon}(\lambda) = \frac{\boldsymbol{P}_{\mathrm{odd}}(\lambda)}{\lambda} + \epsilon\frac{\boldsymbol{P}_{\mathrm{even}}(\lambda)}{\lambda}.$$

It is clear that when $\epsilon = 1$ the nonzero polynomial eigenvalues of $\boldsymbol{R}^1(\lambda)$ coincide with those of the original polynomial $\boldsymbol{P}_{2\ell+1}(\lambda)$. The polynomial $\boldsymbol{P}_{\mathrm{odd}}(\lambda)/\lambda$ is a polynomial of degree 2ℓ with even powers only, whereas the other part has a simple pole at $\lambda = 0$, and all of the powers are odd.

Since $\boldsymbol{P}_{\mathrm{odd}}(\lambda)/\lambda$ is of degree 2ℓ, and since the origin is a pole, for small ϵ the perturbed polynomial will only catch $2\ell n$ of the total $(2\ell+1)n$ polynomial eigenvalues of the full polynomial: the other polynomial eigenvalues are of $\mathcal{O}(\epsilon)$. Thus, we first describe the situation for the rational polynomial, and then finish up by showing where the other n polynomial eigenvalues are located. First consider the full perturbed polynomial,

$$\boldsymbol{P}^{\epsilon}(\lambda) = \boldsymbol{P}_{\text{odd}}(\lambda) + \epsilon \boldsymbol{P}_{\text{even}}(\lambda).$$

Following previous arguments it is the case that if $\boldsymbol{P}_{\text{even}}(\mathrm{i}\mu)$ is definite for all $\mu \in \mathbb{R}$, then there will be no purely imaginary polynomial eigenvalues for any $\epsilon > 0$. Of course, this result also holds true for the rational polynomial $\boldsymbol{R}^{\epsilon}(\lambda)$.

Now, it is clearly the case that $\boldsymbol{R}^{\epsilon}$ and $\boldsymbol{P}^{\epsilon}$ share the same spectrum for any nonzero polynomial eigenvalues which are $\mathcal{O}(1)$; in particular, this holds when $\epsilon = 0$. Since the setup is now essentially the same as for the case of d even, we can just give the highlights which lead to the main result. First, there will be no purely imaginary polynomial eigenvalues for $\epsilon > 0$ if it is assumed that the matrix $\boldsymbol{P}_{\text{even}}(\mathrm{i}\mu)$ is definite for all $\mu \in \mathbb{R}$. The perturbation expansion similar to that which leads to (4.5) yields instead

$$\det\left(\lambda_1\left[-\mathrm{i}\lambda_0 \boldsymbol{P}'_{\text{odd}}(\mathrm{i}\lambda_0)\right]\big|_{E_{\mathrm{i}\lambda_0}} + \left[-\boldsymbol{P}_{\text{even}}(\mathrm{i}\lambda_0)\right]\big|_{E_{\mathrm{i}\lambda_0}}\right) = 0.$$

The calculation follows from the fact that when $\lambda = \mathrm{i}\lambda_0$ is a polynomial eigenvalue with associated eigenvector $\boldsymbol{v}_0$,

$$\left[\frac{\mathrm{d}}{\mathrm{d}\lambda}\left(\frac{\boldsymbol{P}_{\text{odd}}(\lambda)}{\lambda}\right)\bigg|_{\lambda=\mathrm{i}\lambda_0}\right]\boldsymbol{v}_0 = \left[\frac{\boldsymbol{P}'_{\text{odd}}(\mathrm{i}\lambda_0)}{\mathrm{i}\lambda_0}\right]\boldsymbol{v}_0.$$

If $\boldsymbol{P}_{\text{even}}(\mathrm{i}\lambda_0)$ is negative definite for any λ_0, then it will be the case that $k_{\mathrm{i}}^{-}(\mathrm{i}\lambda_0)$ of the polynomial eigenvalues originally at $\mathrm{i}\lambda_0$ move into the right-half plane upon perturbation, whereas if $\boldsymbol{P}_{\text{even}}(\mathrm{i}\lambda_0)$ is positive definite $k_{\mathrm{i}}^{+}(\mathrm{i}\lambda_0)$ of the polynomial eigenvalues originally at $\mathrm{i}\lambda_0$ will move into the right-half plane. From this point forward the argument is exactly the same, and we now know that for the $2\ell n$ polynomial eigenvalues which are $\mathcal{O}(1)$ for small ϵ,

$$\boldsymbol{P}_{\text{even}}(\mathrm{i}\mu) < 0 \quad \Rightarrow \quad k_{\mathrm{r}} + k_{\mathrm{c}} = \mathrm{n}(\boldsymbol{M}_1) + \mathrm{n}\left((-1)^{\ell-1}\boldsymbol{M}_{2\ell+1}\right) + (\ell-1)n,$$

and

$$\boldsymbol{P}_{\text{even}}(\mathrm{i}\mu) > 0 \quad \Rightarrow \quad k_{\mathrm{r}} + k_{\mathrm{c}} = \mathrm{p}(\boldsymbol{M}_1) + \mathrm{p}\left((-1)^{\ell-1}\boldsymbol{M}_{2\ell+1}\right) + (\ell-1)n.$$

Now let us consider the fate of the remaining n polynomial eigenvalues by considering the polynomial $\boldsymbol{P}^{\epsilon}(\lambda)$. When $\epsilon = 0$, $\lambda = 0$ is an algebraically simple polynomial eigenvalue of multiplicity n. Consequently, for small ϵ the small polynomial eigenvalues have the expansion $\lambda = \epsilon\lambda_1 + \mathcal{O}(\epsilon^2)$, and the n associated eigenvectors have the expansion $\boldsymbol{v}_j = \boldsymbol{e}_j + \mathcal{O}(\epsilon)$ for $j = 1, \ldots, n$, where $\boldsymbol{e}_j$ is the jth unit vector. The correction to the polynomial eigenvalue satisfies the characteristic equation

$$\det(\lambda_1 \boldsymbol{M}_1 + \boldsymbol{M}_0) = 0.$$

The condition that $\boldsymbol{P}_{\text{even}}(\mathrm{i}\mu)$ be definite for all μ implies that $\boldsymbol{M}_0$ is definite. If $\boldsymbol{M}_0$ is negative definite, then there will be $\mathrm{p}(\boldsymbol{M}_1)$ real positive zeros of the characteristic equation, and the other zeros will be real and negative. If $\boldsymbol{M}_0$ is positive definite, then there will be $\mathrm{n}(\boldsymbol{M}_1)$ real positive zeros of the characteristic equation, and again the other zeros will be real and negative. This relationship of the polynomial eigenvalues with respect to the imaginary axis holds for all $\epsilon > 0$. Since $\mathrm{n}(\boldsymbol{M}_1) + \mathrm{p}(\boldsymbol{M}_1) = n$, we can now conclude the following regarding the polynomial eigenvalues for the full polynomial.

Lemma 4.7. *Consider the Hermitian polynomial of Lemma 4.1 with $d = 2\ell + 1$. Suppose that all of the purely imaginary polynomial eigenvalues for the ($\star$-even) polynomial $\boldsymbol{P}_{\text{odd}}(\lambda)/\lambda$ are algebraically simple. If for all $\mu \in \mathbb{R}$, $\boldsymbol{P}_{\text{even}}(\mathrm{i}\mu) < 0$, then the instability index for the Hermitian matrix polynomial is*

$$k_{\mathrm{r}} + k_{\mathrm{c}} = \mathrm{n}\left((-1)^{\ell-1}\boldsymbol{M}_{2\ell+1}\right) + \ell n,$$

while if $\boldsymbol{P}_{\text{even}}(\mathrm{i}\mu) > 0$, *the instability index is*

$$k_{\mathrm{r}} + k_{\mathrm{c}} = \mathrm{p}\big((-1)^{\ell-1}\boldsymbol{M}_{2\ell+1}\big) + \ell n.$$

Remark 4.8. If $\boldsymbol{M}_0$ is negative definite in Lemma 4.7, then from Lemma 4.1 the lower bound on the number of positive real-valued polynomial eigenvalues is

$$k_{\mathrm{r}} \geqslant n - \mathrm{n}(\boldsymbol{M}_{2\ell+1}) = \mathrm{p}(\boldsymbol{M}_{2\ell+1}).$$

On the other hand, if $\boldsymbol{M}_0$ is positive definite, the lower bound becomes

$$k_{\mathrm{r}} \geqslant \mathrm{n}(\boldsymbol{M}_{2\ell+1}).$$

Thus, in the cases that a definitive statement can be made regarding the number of polynomial eigenvalues with positive real part, the lower bound on the number of positive real-valued polynomial eigenvalues only depends upon the definiteness of the matrix associated with the highest-degree term.

4.4. Case study: Singulary perturbed Hermitian polynomials

The results of Lemma 4.5 and Lemma 4.7 are valid under the assumption that there are no purely imaginary polynomial eigenvalues. We now show that this assumption can be relaxed if we instead assume that some of the matrices are small in norm relative to the others. We do not do a comprehensive theory here; instead, we focus only on a representative example, and leave other cases to the interested reader. The example will be composed of a perturbation of the well-understood quadratic polynomial.

Consider a singulary perturbed cubic polynomial of the form

$$\boldsymbol{P}_3^{\epsilon}(\lambda) = \boldsymbol{M}_0 + \lambda\boldsymbol{M}_1 + \lambda^2\boldsymbol{M}_2 + \epsilon\lambda^3\boldsymbol{M}_3, \quad 0 < \epsilon \ll 1. \tag{4.6}$$

There will be $2n$ polynomial eigenvalues which are $\mathcal{O}(1)$, and the other n polynomial eigenvalues will be $\mathcal{O}(1/\epsilon)$. The $\mathcal{O}(1)$ polynomial eigenvalues are to leading order found as polynomial eigenvalues to the quadratic polynomial

$$\boldsymbol{P}_2^0(\lambda) = \boldsymbol{M}_0 + \lambda\boldsymbol{M}_1 + \lambda^2\boldsymbol{M}_2,$$

and the other n polynomial eigenvalues are found by locating the polynomial eigenvalues for the linear polynomial

$$\boldsymbol{P}_1^0(z) = \boldsymbol{M}_2 + z\boldsymbol{M}_3, \quad z := \lambda/\epsilon.$$

If we assume that $\boldsymbol{M}_1$ is definite, and if we also assume that the purely imaginary eigenvalues for the $\star$-even polynomial $\boldsymbol{M}_0 + \lambda^2\boldsymbol{M}_2$ are algebraically simple, then we can invoke Corollary 4.6 to say that for the polynomial $\boldsymbol{P}_2^0(\lambda)$,

$$k_{\mathrm{r}} \geqslant \big|\mathrm{n}(\boldsymbol{M}_2) - \mathrm{n}(\boldsymbol{M}_0)\big|, \qquad k_{\mathrm{r}} + k_{\mathrm{c}} = \begin{cases} \mathrm{n}(\boldsymbol{M}_0) + \mathrm{n}(\boldsymbol{M}_2), & \boldsymbol{M}_1 > 0, \\ \mathrm{p}(\boldsymbol{M}_0) + \mathrm{p}(\boldsymbol{M}_2), & \boldsymbol{M}_1 < 0. \end{cases}$$

Now consider the linear polynomial $\boldsymbol{P}_1^0(z)$. Without any further assumptions we can invoke Lemma 4.1 to say that

$$k_{\mathrm{r}} \geqslant \big|\mathrm{n}(\boldsymbol{M}_3) - \mathrm{n}(\boldsymbol{M}_2)\big|.$$

Thus, if $\epsilon > 0$ is sufficiently small we have for the full polynomial the refined instability criterion

$$k_{\mathrm{r}} \geqslant \big|\mathrm{n}(\boldsymbol{M}_3) - \mathrm{n}(\boldsymbol{M}_2)\big| + \big|\mathrm{n}(\boldsymbol{M}_2) - \mathrm{n}(\boldsymbol{M}_0)\big| \geqslant \big|\mathrm{n}(\boldsymbol{M}_3) - \mathrm{n}(\boldsymbol{M}_0)\big|,$$

where the latter inequality follows from the triangle inequality. Thus, when compared to Lemma 4.1 for the full polynomial we have a (potentially) larger lower bound on the total number of positive real polynomial eigenvalues.

A more concrete result concerning the distribution of polynomial eigenvalues for $\boldsymbol{P}_1^0(z)$ cannot be made without a definiteness assumption on one of the two coefficient matrices. If (at least) one of the matrices is assumed to be definite, then all of the polynomial eigenvalues will be purely real ($k_\mathrm{c} = 0$). If $\boldsymbol{M}_3$ is definite, then the result is

$$k_\mathrm{r} = \begin{cases} \mathrm{n}(\boldsymbol{M}_2), & \boldsymbol{M}_3 > 0, \\ \mathrm{p}(\boldsymbol{M}_2), & \boldsymbol{M}_3 < 0, \end{cases}$$

whereas if $\boldsymbol{M}_2$ is definite,

$$k_\mathrm{r} = \begin{cases} \mathrm{n}(\boldsymbol{M}_3), & \boldsymbol{M}_2 > 0, \\ \mathrm{p}(\boldsymbol{M}_3), & \boldsymbol{M}_2 < 0. \end{cases}$$

Combining all of these results gives the following result.

Lemma 4.9. *Consider the singularly perturbed cubic polynomial $\boldsymbol{P}_3^\epsilon(\lambda)$ for $0 < \epsilon \ll 1$ sufficiently small. The number of real positive polynomial eigenvalues has the lower bound*

$$k_\mathrm{r} \geqslant \underbrace{\left|\mathrm{n}(\boldsymbol{M}_3) - \mathrm{n}(\boldsymbol{M}_2)\right|}_{\mathcal{O}(1/\epsilon)} + \underbrace{\left|\mathrm{n}(\boldsymbol{M}_2) - \mathrm{n}(\boldsymbol{M}_0)\right|}_{\mathcal{O}(1)},$$

where the label under the brace refers to the size of these polynomial eigenvalues. Henceforth assume that $\boldsymbol{M}_1$ is definite. Start with the assumption that $\boldsymbol{M}_1 > 0$. If $\boldsymbol{M}_2$ is definite, then for the total number of unstable polynomial eigenvalues we have

$$k_\mathrm{r} = \mathrm{n}(\boldsymbol{M}_0) + \mathrm{n}(\boldsymbol{M}_3), \qquad k_\mathrm{c} = 0; \quad \boldsymbol{M}_2 > 0$$
$$k_\mathrm{r} + k_\mathrm{c} = \mathrm{n}(\boldsymbol{M}_0) + \mathrm{p}(\boldsymbol{M}_3) + n; \qquad \boldsymbol{M}_2 < 0,$$

while if $\boldsymbol{M}_3$ is definite we have

$$k_\mathrm{r} + k_\mathrm{c} = \begin{cases} \mathrm{n}(\boldsymbol{M}_0) + 2\mathrm{n}(\boldsymbol{M}_2); & \boldsymbol{M}_3 > 0, \\ \mathrm{n}(\boldsymbol{M}_0) + n; & \boldsymbol{M}_3 < 0. \end{cases}$$

Now assume that $\boldsymbol{M}_1 < 0$. If $\boldsymbol{M}_2$ is definite, then for the total number of unstable polynomial eigenvalues we have

$$k_\mathrm{r} + k_\mathrm{c} = \mathrm{p}(\boldsymbol{M}_0) + \mathrm{n}(\boldsymbol{M}_3) + n; \qquad \boldsymbol{M}_2 > 0,$$
$$k_\mathrm{r} = \mathrm{p}(\boldsymbol{M}_0) + \mathrm{p}(\boldsymbol{M}_3), \qquad k_\mathrm{c} = 0; \quad \boldsymbol{M}_2 < 0,$$

while if $\boldsymbol{M}_3$ is definite we have

$$k_\mathrm{r} + k_\mathrm{c} = \begin{cases} \mathrm{p}(\boldsymbol{M}_0) + n; & \boldsymbol{M}_3 > 0, \\ \mathrm{p}(\boldsymbol{M}_0) + 2\mathrm{p}(\boldsymbol{M}_2); & \boldsymbol{M}_3 < 0. \end{cases}$$

Proof. We only need to consider the two cases in which $\boldsymbol{M}_2$ has the same definiteness as $\boldsymbol{M}_1$, as the other results follow from the discussion preceding the statement of the lemma. Suppose that $\boldsymbol{M}_1, \boldsymbol{M}_2 > 0$. For the $\mathcal{O}(1)$ polynomial eigenvalues we have

$$k_\mathrm{r} \geqslant \mathrm{n}(\boldsymbol{M}_0), \qquad k_\mathrm{r} + k_\mathrm{c} = \mathrm{n}(\boldsymbol{M}_0) \quad \Rightarrow \quad k_\mathrm{r} = \mathrm{n}(\boldsymbol{M}_0), \qquad k_\mathrm{c} = 0.$$

For the $\mathcal{O}(1/\epsilon)$ polynomial eigenvalues we have $k_\mathrm{r} = \mathrm{n}(\boldsymbol{M}_3)$ with $k_\mathrm{c} = 0$. In conclusion, by summing we get the desired result. If $\boldsymbol{M}_1, \boldsymbol{M}_2 < 0$, the same result follows upon substituting $\mathrm{n}(\cdot)$ with $\mathrm{p}(\cdot)$. □

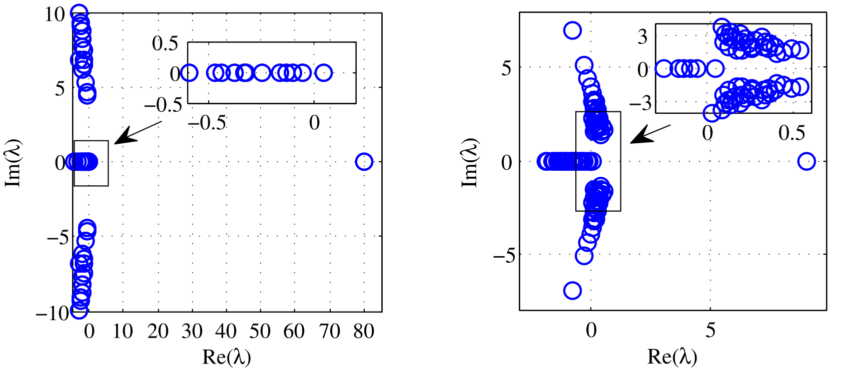

Fig. 3. (Color online.) A numerical demonstration of Lemma 4.9 when $n = 32$, $\mathrm{n}(\boldsymbol{M}_0) = \mathrm{n}(\boldsymbol{M}_3) = 1$, and $\boldsymbol{M}_1, \boldsymbol{M}_2 > 0$. The prediction is that for $0 < \epsilon \ll 1$ there will be two real and positive polynomial eigenvalues, and all of the other polynomial eigenvalues will have negative real part. This is precisely what is seen in the left panel, where $\epsilon = 0.01$. In the right panel $\epsilon = 0.25$, and it is clear that ϵ is too large for Lemma 4.9 to be valid.

It is interesting to compare the perturbative result of Lemma 4.9 with the nonperturbative result of Lemma 4.7. For example, suppose that $\boldsymbol{M}_0 < 0$ and $\boldsymbol{M}_2 > 0$, so that the instability index becomes

$$k_{\mathrm{r}} + k_{\mathrm{c}} = \mathrm{n}(\boldsymbol{M}_3) + n.$$

This result is valid for all $\epsilon > 0$. Furthermore, the lower bound on the number of real polynomial eigenvalues guarantees that $k_{\mathrm{c}} = 0$ for small ϵ, so that in this case all of the unstable polynomial eigenvalues are real-valued. As we saw in Remark 4.8, the lower bound becomes $k_{\mathrm{r}} \geqslant \mathrm{p}(\boldsymbol{M}_3)$ once ϵ is sufficiently large. We see from Lemma 4.9 that if $\epsilon > 0$ is now small, and if we relax the restriction $\boldsymbol{M}_0 < 0$ and allow $\boldsymbol{M}_0$ to have some positive directions, then using $\mathrm{n}(\boldsymbol{M}_0) = n - \mathrm{p}(\boldsymbol{M}_0)$ the result becomes

$$k_{\mathrm{r}} + k_{\mathrm{c}} = \begin{cases} \mathrm{n}(\boldsymbol{M}_3) + n + \mathrm{p}(\boldsymbol{M}_0), & \boldsymbol{M}_1 < 0, \\ \mathrm{n}(\boldsymbol{M}_3) + n - \mathrm{p}(\boldsymbol{M}_0), & \boldsymbol{M}_1 > 0, \end{cases} \qquad k_{\mathrm{r}} \geqslant \mathrm{n}(\boldsymbol{M}_3) + n - \mathrm{p}(\boldsymbol{M}_0).$$

This yields the upper bound $k_{\mathrm{c}} \leqslant 2\mathrm{p}(\boldsymbol{M}_0)$ for $\boldsymbol{M}_1 < 0$, with $k_{\mathrm{c}} = 0$ if $\boldsymbol{M}_1 > 0$. On the other hand, if $\boldsymbol{M}_0 > 0$ and $\boldsymbol{M}_2 < 0$ the index valid for all $\epsilon > 0$ becomes

$$k_{\mathrm{r}} + k_{\mathrm{c}} = \mathrm{p}(\boldsymbol{M}_3) + n, \quad k_{\mathrm{r}} \geqslant \mathrm{n}(\boldsymbol{M}_3).$$

If $\epsilon > 0$ is sufficiently small, then relaxing the condition that $\boldsymbol{M}_0$ be definite gives

$$k_{\mathrm{r}} + k_{\mathrm{c}} = \begin{cases} \mathrm{p}(\boldsymbol{M}_3) + n - \mathrm{n}(\boldsymbol{M}_0), & \boldsymbol{M}_1 < 0, \\ \mathrm{p}(\boldsymbol{M}_3) + n + \mathrm{n}(\boldsymbol{M}_0), & \boldsymbol{M}_1 > 0, \end{cases} \qquad k_{\mathrm{r}} \geqslant \mathrm{p}(\boldsymbol{M}_3) + n - \mathrm{n}(\boldsymbol{M}_0).$$

If $\boldsymbol{M}_1 < 0$ all of the unstable polynomial eigenvalues are real-valued; otherwise, there is the upper bound $k_{\mathrm{c}} \leqslant 2\mathrm{n}(\boldsymbol{M}_0)$.

For a numerical demonstration of Lemma 4.9, consider Fig. 3. For this figure the polynomial eigenvalues when $n = 32$, $\mathrm{n}(\boldsymbol{M}_0) = \mathrm{n}(\boldsymbol{M}_3) = 1$, and $\boldsymbol{M}_1, \boldsymbol{M}_2 > 0$ are calculated using the MATLAB command "polyeig". In the left panel $\epsilon = 0.01$, and in the right panel $\epsilon = 0.25$. The result of Lemma 4.9 predicts that for $\epsilon > 0$ sufficiently small there will be one real and positive polynomial eigenvalue of $\mathcal{O}(1)$, and one real and positive polynomial eigenvalue of $\mathcal{O}(1/\epsilon)$. Furthermore, there will be no other polynomial eigenvalues with positive real part. This prediction is clearly validated. On the other hand, the result is clearly no longer applicable in the right panel, where $\epsilon = 0.25$.

Acknowledgements

T.K. gratefully acknowledges the support of the Jack and Lois Kuipers Applied Mathematics Endowment, a Calvin Research Fellowship, and the National Science Foundation under grant DMS-1108783.

The research of E.H., H.-P.K., and J.T. was supported by the National Science Foundation under grant DMS-1108783.

References

[1] M. Al-Ammari, Analysis of structured polynomial eigenvalue problems, PhD thesis, The University of Manchester, 2011.
[2] M. Al-Ammari, F. Tisseur, Standard triples of structured matrix polynomials, Linear Algebra Appl. 437 (2012) 817–834.
[3] T. Azizov, M. Chugunova, Pontryagin's theorem and spectral stability analysis of solitons, Math. Notes 86 (5) (2009) 612–624.
[4] T. Azizov, I. Iokhvidov, Linear Operators in Spaces with an Indefinite Metric, Wiley, 1989.
[5] J. Bronski, M. Johnson, T. Kapitula, An instability index theory for quadratic pencils and applications, preprint, 2012.
[6] M. Chugunova, D. Pelinovsky, On quadratic eigenvalue problems arising in stability of discrete vortices, Linear Algebra Appl. 431 (2009) 962–973.
[7] M. Chugunova, D. Pelinovsky, Count of eigenvalues in the generalized eigenvalue problem, J. Math. Phys. 51 (5) (2010) 052901.
[8] E. Frank, On the zeros of polynomials with complex coefficients, Bull. Amer. Math. Soc. 52 (1946) 144–158.
[9] I. Gohberg, P. Lancaster, L. Rodman, Indefinite Linear Algebra and Applications, Birkhäuser, 2005.
[10] I. Gohberg, P. Lancaster, L. Rodman, Matrix Polynomials, Classics Appl. Math., vol. 58, SIAM, 2009.
[11] M. Grillakis, Linearized instability for nonlinear Schrödinger and Klein–Gordon equations, Comm. Pure Appl. Math. 46 (1988) 747–774.
[12] D. Henrion, O. Bachelier, M. Šebek, $\mathcal{D}$-stability of polynomial matrices, preprint, 2012.
[13] N. Higham, F. Tisseur, Bounds for eigenvalues of matrix polynomials, Linear Algebra Appl. 358 (2003) 5–22.
[14] M. Hărăguş, T. Kapitula, On the spectra of periodic waves for infinite-dimensional Hamiltonian systems, Phys. D 237 (20) (2008) 2649–2671.
[15] C.K.R.T. Jones, Instability of standing waves for non-linear Schrödinger-type equations, Ergodic Theory Dynam. Systems 8 (1988) 119–138.
[16] C.K.R.T. Jones, J. Moloney, Instability of standing waves in nonlinear optical waveguides, Phys. Lett. A 117 (1986) 175–180.
[17] T. Kapitula, K. Promislow, Stability indices for constrained self-adjoint operators, Proc. Amer. Math. Soc. 140 (3) (2012) 865–880.
[18] T. Kapitula, K. Promislow, Spectral and Dynamical Stability of Nonlinear Waves, Springer-Verlag, 2013, in press.
[19] T. Kato, Perturbation Theory for Linear Operators, Springer-Verlag, Berlin, 1980.
[20] R. Kollár, Homotopy method for nonlinear eigenvalue pencils with applications, SIAM J. Math. Anal. 43 (2) (2011) 612–633.
[21] R. Kollár, P. Miller, Graphical Krein signature theory and Evans–Krein functions, preprint, 2013.
[22] A. Kostenko, On the defect index of quadratic self-adjoint operator pencils, Math. Notes 72 (2) (2002) 285–290.
[23] F. Kraus, M. Mansour, M. Šebek, Hurwitz matrix for polynomial matrices, in: R. Jeltsch, M. Mansour (Eds.), Stability Theory. Proceedings of the Conference – Centennial Hurwitz on Stability Theory, in: Internat. Ser. Numer. Math., vol. 121, 1996, pp. 67–74.
[24] M. Krein, A generalization of some investigations of A.M. Lyapunov on linear differential equations with periodic coefficients, Dokl. Akad. Nauk SSSR 73 (1950).
[25] M. Krein, On the application of an algebraic proposition in the theory of matrices of monodromy, Uspekhi Mat. Nauk 6 (1) (1951) 171–177.
[26] M. Krein, G. Kjubarskii, Analytic properties of multipliers of periodic canonical differential systems of positive type, Amer. Math. Soc. Transl. 89 (1970) 1–28.
[27] D. Mackey, N. Mackey, C. Mehl, V. Mehrmann, Structured polynomial eigenvalue problems: good vibrations from good linearizations, SIAM J. Matrix Anal. Appl. 28 (4) (2006) 1029–1051.
[28] D. Mackey, N. Mackey, C. Mehl, V. Mehrmann, Structures of alternating matrix polynomials, Linear Algebra Appl. 432 (4) (2010) 867–891.
[29] N. Mackey, D. Mackey, C. Mehl, V. Mehrmann, Möbius Transformations of Matrix Polynomials, ICIAM, 2011.
[30] A. Markus, Introduction to the Spectral Theory of Polynomial Operator Pencils, Transl. Math. Monogr., vol. 71, Amer. Math. Soc., 1988.
[31] V. Mehrmann, D. Watkins, Polynomial eigenvalue problems with Hamiltonian structure, Electron. Trans. Numer. Anal. 13 (2002) 106–118.
[32] D. Pelinovsky, Inertia law for spectral stability of solitary waves in coupled nonlinear Schrödinger equations, Proc. R. Soc. Lond. Ser. A Math. Phys. Eng. Sci. 461 (2005) 783–812.
[33] L. Pontryagin, Hermitian operators in spaces with indefinite metric, Izv. Akad. Nauk SSSR Ser. Mat. 8 (1944) 243–280.
[34] A. Shkalikov, Operator pencils arising in elasticity and hydrodynamics: the instability index formula, in: I. Gohberg, P. Lancaster, P. Shivakumar (Eds.), Recent Developments in Operator Theory and its Applications, in: Oper. Theory Adv. Appl., vol. 87, 1996, pp. 358–385.
[35] A. Shkalikov, Dissipative operators in the Krein space. Invariant subspaces and properties of restrictions, Funct. Anal. Appl. 41 (2) (2007) 154–167.