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On the rigidity of the cotangent complex at the prime 2[☆]

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ABSTRACT

In [D. Quillen, On the (co)homology of commutative rings, Proc. Symp. Pure Math. 17 (1970) 65–87; L. Avramov, Locally complete intersection homomorphisms and a conjecture of Quillen on the vanishing of cotangent homology, Annals of Math. 2 (150) (1999) 455–487] a conjecture was posed to the effect that if $R \to A$ is a homomorphism of Noetherian commutative rings then the flat dimension, as defined in the derived category of A-modules, of the associated cotangent complex $\mathbb{L}_{A/R}$ satisfies: $\mathrm{fd}_A\, \mathbb{L}_{A/R} < \infty \Longrightarrow \mathrm{fd}_A\, \mathbb{L}_{A/R} \leq 2$. The aim of this paper is to initiate an approach for solving this conjecture when R has characteristic 2 using simplicial algebra techniques. To that end, we obtain two results. First, we prove that the conjecture can be reframed in terms of certain nilpotence properties for the divided square γ_2 and the André operation ϑ as it acts on $\mathrm{Tor}_R(A, \ell)$, ℓ any residue field of A. Second, we prove the conjecture is valid in two cases: when $\mathrm{fd}_R\, A < \infty$ and when R is a Cohen–Macaulay ring.

0. Introduction

In [1,2], a notion of relative homology $D_*(A|R; M)$ was defined and studied for a homomorphism $R \to A$ of commutative rings with coefficients in A-modules. These are defined as the derived functor of abelianization on the homotopy category of simplicial commutative R-algebras. Thus they can be viewed in terms of the Kahler differentials:

$$D_*(A|R; M) := \pi_*(\Omega_{X./R} \otimes_{X.} M) = H_*(\mathbb{L}_{A/R} \otimes_A M)$$

where $X.$ is a cofibrant replacement of A in Quillen's simplicial model structure for simplicial commutative R-algebras (cf. [3]) and

$$\mathbb{L}_{A/R} := N(\Omega_{X./R} \otimes_{X.} A)$$

is the *cotangent complex* of $R \to A$, which can be viewed as an object of the derived category of A-modules. Thus various homological dimensions can be attached to it, such as the projective or flat dimension. In [2], Quillen conjectured certain rigidity properties regarding the cotangent complex. In [4], these conjectures were framed as follows:

Quillen's Conjecture. *Let $R \to A$ be a homomorphism of Noetherian rings such that $\mathrm{fd}_A\, \mathbb{L}_{A/R} < \infty$. Then*

(1) $\mathrm{fd}_A\, \mathbb{L}_{A/R} \leq 2$*;*
(2) *If, in addition, $\mathrm{fd}_R\, A < \infty$, then $\mathrm{fd}_A\, \mathbb{L}_{A/R} \leq 1$. Furthermore, $R \to A$ is a locally complete intersection homomorphism.*

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Much is currently known about the validity of this conjecture. In [5], T. Gulliksen showed the validity of (2) for a local ring $R \to \ell$, with ℓ a rational field, utilizing Quillen's result on the collapse of the fundamental spectral sequence. Subsequently, M. André [6] showed that this approach fails when ℓ is a primary field. In [7], L. Avramov and S. Halperin established part (2) when R contains a rational field. Later, the author [8] gave a proof of the first part of (2) when A contains a primary field. Finally, Avramov [4] gave a complete solution to (2) while simultaneously establishing general properties for locally complete intersection homomorphisms. Following that work, Avramov and S. Iyengar [9] proved part (1) for $R \to A$ an algebra retract. The general case of (1) is still open. See [10,11] for an excellent account of the history of Quillen's Conjecture and the current state of affairs.

The purpose to this paper is to make a further contribution to resolving Quillen's Conjecture by establishing our:

Theorem A (*Main Theorem*). *Let $R \to S$ be a homomorphism of Noetherian rings with $\mathrm{fd}_A\, \mathbb{L}_{A/R} < \infty$. Then*

(1) $\mathrm{fd}_A\, \mathbb{L}_{A/R} \leq 2$ *if R is a Cohen–Macaulay ring of characteristic 2;*
(2) $\mathrm{fd}_A\, \mathbb{L}_{A/R} \leq 1$ *if $\mathrm{fd}_R\, A < \infty$ and A has characteristic 2.*

Note: The second part of Theorem A is true independent of characteristic. This is the main result of L. Avramov in [4].

Philosophically, the way we will approach proving the Main Theorem is by building off of André's observations in [12,6]. The restriction in characteristic allows us to take advantage of both the methods the author used in [8,13] together with the methods of P. Goerss in [14]. In both places, the properties and internal structure of the homotopy and homology of simplicial commutative algebras over a field of characteristic 2 are analyzed and used. In particular, the higher divided squares of W. Dwyer [15] act on the homotopy groups, with two such operations being the *divided square* γ_2 and the *André operation* ϑ [6]. In the context of homomorphisms $R \to (A, \ell)$ of local rings, these operations act as follows:

$$\gamma_2 : \mathrm{Tor}_n^R(A, \ell) \to \mathrm{Tor}_{2n}^R(A, \ell)$$

and

$$\vartheta : \mathrm{Tor}_n^R(A, \ell) \to \mathrm{Tor}_{2n-1}^R(A, \ell).$$

In this context, we will define $R \to (S, \ell)$ to be:

(1) *γ_2-nilpotent* provided that for each $x \in \mathrm{Tor}_{\geq 2}^R(A, \ell)$ there is an $n > 0$ such that $\gamma_2^n(x) = 0$;
(2) *André nilpotent* provided that for each $x \in \mathrm{Tor}_{\geq 3}^R(A, \ell)$ there is an $n > 0$ such that $\vartheta^n(x) = 0$.

In connection with proving the Main Theorem, we will establish:

Theorem B. *Let $R \to (A, \ell)$ be a surjective homomorphism of local rings with $\mathrm{char}\, \ell = 2$ and $\mathrm{fd}_A\, \mathbb{L}_{A/R} < \infty$. Then*

(1) *$D_s(A/R; \ell) = 0$ for all $s > 2$ if and only if $R \to (A, \ell)$ is André nilpotent;*
(2) *$D_s(A/R; \ell) = 0$ for all $s > 1$ if and only if $R \to (A, \ell)$ is γ_2-nilpotent.*

Notes:

(1) For both parts of this theorem, the proof of necessity is straightforward. For (1), if $\mathrm{fd}_A\, \mathbb{L}_{A/R} \leq 2$ then it follows that $\mathrm{Tor}_*^R(A, \ell)$ is a free divided power algebra and so André nilpotency follows from $\vartheta\gamma_2 = 0$. For (2), if $\mathrm{fd}_A\, \mathbb{L}_{A/R} \leq 1$ then it follows that $\mathrm{Tor}_*^R(A, \ell)$ is finite graded and, hence, γ_2-nilpotent.
(2) Theorem B(2) gives a generalization of Quillen's Conjecture (2) (for local rings of characteristic 2) as it allows for cases of Tor-modules which are non-trivial for infinitely many degrees.

The simplicial setting

In [8,13], the author took a different tack to proving part (2). By noting that the cotangent complex is defined for an arbitrary simplicial commutative R-algebra A, an extension of part (2) to such A was proved when A has the following properties: char $\pi_0 A > 0$ and A has *strongly finite Noetherian homotopy*, i.e. $\pi_0 A$ is Noetherian, $\pi_* A$ is a finite graded $\pi_0 A$-module and $\mathrm{fd}_{\pi_0 A}\, \pi_* A < \infty$. ($A$ will be said to have *finite Noetherian homotopy* if this last condition is dropped.) This enabled the author to use simplicial methods. It is the aim of this paper to use these same methods to approach part (1).

Now, assume the induced map $R \to \pi_0 A$ is a surjection. We define A to be a *a locally homotopy n-intersection*, where n is a natural number, provided for each $\mathfrak{p} \in \mathrm{Spec}(\pi_0 A)$ the *connected component at* $\mathfrak{p}$

$$A(\mathfrak{p}) := A \otimes_R^{\mathbf{L}} k(\mathfrak{p}) \simeq S_{k(\mathfrak{p})}(W)$$

with W a connected simplicial $k(\mathfrak{p})$-module satisfying $\pi_s W = 0$ for $s > n$. Here and throughout $S_{k(\mathfrak{p})}(-)$ denotes the free commutative $k(\mathfrak{p})$-algebra functor.

Notes: Let A be a simplicial commutative algebra with strongly finite Noetherian homotopy.

(1) In fact, a definition of locally homotopy n-intersection can be made when $R \to \pi_0 A$ is a general homomorphism of Noetherian rings using the methods of [13].

(2) In [13], a locally homotopy 1-intersection is called a *homotopy complete intersection* shown to be equivalent to $\mathrm{fd}_A\ \mathbb{L}_{A/R} \leq 1$.
(3) In [16], a notion of *homotopy Gorenstein* is defined for such A and was shown to be a property of A when $\mathrm{fd}_A\ \mathbb{L}_{A/R} < \infty$, independent of the characteristic of $\pi_0 A$.

Within this simplicial context, we will be able to establish Theorem B by first proving:

Theorem C. *Let A be a simplicial commutative R-algebra with finite Noetherian homotopy such that $R \to \pi_0 A$ is a surjection. Let $\mathfrak{p} \in \mathrm{Spec}(\pi_0 A)$ be such that $\mathrm{char}\ k(\mathfrak{p}) = 2$ and $D_s(A|R; k(\mathfrak{p})) = 0$ for $s \gg 0$. Then:*
(1) *$A(\mathfrak{p})$ is André nilpotent if and only if $A(\mathfrak{p})$ is a homotopy 2-intersection;*
(2) *$A(\mathfrak{p})$ is γ_2-nilpotent if and only if $A(\mathfrak{p})$ is a homotopy 1-intersection.*

As a consequence of the Theorem C, given its assumptions, if $D_s(A|R; k(\mathfrak{p})) = 0$ for $s > 2$ then $A(\mathfrak{p})$ is André nilpotent. From this we offer the following:

Nilpotence Conjecture. *Let A be a simplicial commutative R-algebra with finite Noetherian homotopy such that $R \to \pi_0 A$ is a surjection. Let $\mathfrak{p} \in \mathrm{Spec}\ \pi_0 A$ be such that $\mathrm{char}(k(\mathfrak{p})) = 2$. Then $A(\mathfrak{p})$ is André nilpotent if $D_s(A|R; k(\mathfrak{p})) = 0$ for $s \gg 0$.*

Assuming the validity of the Nilpotence Conjecture gives us the following:

Theorem D. *Let $R \to A$ be a homomorphism of Noetherian rings with $\mathrm{char}\ A = 2$ and $\mathrm{fd}_A\ \mathbb{L}_{A/R} < \infty$. If the Nilpotence Conjecture is true then $\mathrm{fd}_A\ \mathbb{L}_{A/R} \leq 2$.*

Proof. Let $f : R \to A$ be the homomorphism of Noetherian rings with $\mathrm{char}\ R = 2$. Let $\mathfrak{q} \in \mathrm{Spec}\ A$ and $\mathfrak{p} = f^{-1}(\mathfrak{q}) \in \mathrm{Spec}\ R$. By the main result of [17], there is a Cohen factorization of complete local rings

$$R_{\widehat{\mathfrak{p}}} \to R' \to A_{\widehat{\mathfrak{q}}}$$

satisfying

(a) $R_{\widehat{\mathfrak{p}}} \to R'$ is a faithfully flat monomorphism with weakly regular fibre;
(b) $R' \to A_{\widehat{\mathfrak{q}}}$ is a surjection.

from which arises

(c) $\mathrm{fd}_A\ \mathbb{L}_{A/R} \leq n$ if and only if $D_s(A_{\widehat{\mathfrak{q}}}/R'; k(\mathfrak{q})) = 0$ for all $s > n$ and for all $\mathfrak{q} \in \mathrm{Spec}\ A$.

See [4, Section 1] for further details regarding (c). Now, by (c) and the Nilpotence Conjecture, $A_{\widehat{\mathfrak{q}}}(\widehat{\mathfrak{q}})$ is André nilpotent. Hence, $R' \to A_{\widehat{\mathfrak{q}}}$ is André nilpotent and it follows that $D_s(A_{\widehat{\mathfrak{q}}}/R'; k(\mathfrak{q})) = 0$ for $s > 2$ by Theorem C.1. Since this holds for all $\mathfrak{q} \in \mathrm{Spec}\ A$, Theorem D follows from (c). □

As a contribution to establishing the Nilpotence Conjecture, we will prove:

Theorem E. *The Nilpotence Conjecture is true when R is a Cohen–Macaulay ring of characteristic 2.*

We close this section by indicating how Theorems A and B follows from Theorems C–E.

Proof of Theorem B. Let $\mathfrak{p}$ be the maximal ideal of the local ring A. Then $\pi_* A(\mathfrak{p}) \cong \mathrm{Tor}^R_*(A, \ell)$. If $\mathrm{fd}_A\ \mathbb{L}_{A/R} < \infty$ then $D_s(A|R; \ell) = 0$ for $s \gg 0$. Thus:

(1) If $A(\mathfrak{p})$ is André nilpotent then $A(\mathfrak{p})$ is a homotopy 2-intersection by Theorem C.1. Thus $A(\mathfrak{p}) \simeq S_\ell(W)$ with $\pi_i W = 0$ for $i > 2$. By flat base change
$$\mathbb{L}_{A/R} \otimes^{\mathbf{L}}_A \ell \cong \mathbb{L}_{A(\mathfrak{p})/\ell} \otimes^{\mathbf{L}}_{A(\mathfrak{p})} \ell \cong W.$$
Thus $D_s(A/R; \ell) = 0$ for all $s > 2$.
(2) If $A(\mathfrak{p})$ is γ_2-nilpotent then $A(\mathfrak{p})$ is a homotopy 1-intersection by Theorem C. Thus $A(\mathfrak{p}) \simeq S_\ell(W)$ with $\pi_i W = 0$ for $i > 1$. Again, by flat base change
$$\mathbb{L}_{A/R} \otimes^{\mathbf{L}}_A \ell \cong \mathbb{L}_{A(\mathfrak{p})/\ell} \otimes^{\mathbf{L}}_{A(\mathfrak{p})} \ell \cong W$$
and it follows that $D_s(A/R; \ell) = 0$ for all $s > 1$.

As noted before, the converses are straightforward. □

Proof of Theorem A. For each $\mathfrak{q} \in \mathrm{Spec}\ A$, choose a Cohen factorization as described above. Then

(1) If R is Cohen–Macaulay then R' is Cohen–Macaulay for each $\mathfrak{q} \in \mathrm{Spec}\ A$ [17]. It follows that $\mathrm{Tor}^{R'}_*(A_{\widehat{\mathfrak{q}}}, k(\mathfrak{q}))$ is André nilpotent, by Theorem E. The result now follows from Theorem B.1 and (c) above. Notice that this result also follows from Theorem D.
(2) If $\mathrm{fd}_R\ A < \infty$ then $\mathrm{fd}_{R'}\ A_{\widehat{\mathfrak{q}}} < \infty$ [17] and so $\mathrm{Tor}^{R'}_*(A(\widehat{\mathfrak{q}}), k(\mathfrak{q}))$ is finite graded and, hence, γ_2-nilpotent. Thus Theorem A.2 follows from Theorem B.2 and (c) above. □

Organization. Since our proofs require technical results pertaining to the homotopy and homology of simplicial commutative algebras in characteristic 2, we begin with a review and extension of the work in [15,14] pertinent to our purposes in the first section. This is followed by two sections which provide proofs of Theorems C and E.

1. Homotopy and homology of simplicial commutative algebras of characteristic 2

The proofs of Theorems C and E relied heavily on several technical aspects of the homotopy and homology of simplicial commutative algebras over a general field of characteristic 2. In this section, we provide an exposition of the aspects of this theory that suit our purposes. We do this for the following reason: the standard references to this theory [15,14] focus on the ground field $\mathbb{F}_2$ and say nothing explicit about general ground fields of characteristic 2. That this should be straightforward should not be a surprise: the analogous result for rational ground fields can be similarly resolved. See [18, Section 4]. It will be our assertion that the known theory over $\mathbb{F}_2$ extends to any field extension without little change, the only important addition being a needed account of how the action of the Frobenius is incorporated. That this is straightforward can already be culled from [15,14]. Furthermore, that this simple addition is all that is required matches with the analogous account for Steenrod operations. See [19, Section 4] and [20].

1.1. Review of the homotopy of simplicial commutative algebras over a field

Let A be a simplicial commutative ℓ-algebra where ℓ is a field of characteristic 2. In this section we review some basic facts about the homotopy groups of such objects, computed as the homotopy groups of simplicial ℓ-modules.

Let $\mathcal{A}_\ell$ be the category of *augmented ℓ-algebras*, i.e. commutative ℓ-algebras augmented over ℓ. Let $s\mathcal{A}_\ell$ be the category of simplicial objects over $\mathcal{A}_\ell$. Then for $A \in s\mathcal{A}_\ell$ and $n \geq 0$ we have a natural isomorphism

$$\pi_n A \cong [S_\ell(n), A]_{\mathrm{Ho}(s\mathcal{A}_\ell)}$$

where $S_\ell(n) = S_\ell(K(n))$ is the free commutative (i.e. symmetric) algebra generated by $K(n)$, $K(n)$ being the simplicial ℓ-module satisfying $\pi_* K(n) \cong \ell$ concentrated in degree n. We will use this relation to determine the natural primary algebra structure on $\pi_* A$.

Given integers $r_1, \ldots, r_m, t_1, \ldots, t_n \neq 0$ an *multioperation* of degree $(r_1, \ldots, r_m; t_1, \ldots, t_n)$ is a natural map

$$\theta : \pi_{r_1} \times \cdots \times \pi_{r_m} \to \pi_{t_1} \times \cdots \times \pi_{t_n}$$

of functors on $s\mathcal{A}_\ell$. Let $\mathrm{Nat}_{r_1,\ldots,r_m;t_1,\ldots,t_n}$ be the set of multioperations of degree $(r_1, \ldots, r_m; t_1, \ldots, t_n)$. It is straightforward to show that

$$\mathrm{Nat}_{r_1,\ldots,r_m;t_1,\ldots,t_n} \cong \mathrm{Nat}_{r_1,\ldots,r_m;t_1} \times \cdots \times \mathrm{Nat}_{r_1,\ldots,r_m;t_n}.$$

Now, we define

$$f : \mathrm{Nat}_{r_1,\ldots,r_m;t} \to \pi_t(S_\ell(r_1) \otimes_\ell \cdots \otimes_\ell S_\ell(r_m)) \tag{1.1}$$

as follows. Let $\mathcal{N} = \mathrm{Nat}_{r_1,\ldots,r_m;t}$ and let $X = S_\ell(r_1) \otimes_\ell \cdots \otimes_\ell S_\ell(r_m)$. For each $1 \leq j \leq m$, let $\iota_j \in \pi_{r_j} X$ be the homotopy class of the inclusion $S_\ell(r_j) \to X$. Given $\theta \in \mathcal{N}$ there is an induced map

$$\theta_X : \pi_{r_1} X \times \cdots \times \pi_{r_m} X \to \pi_t X.$$

Thus we can define $f : \mathcal{N} \to \pi_t X$ by

$$f(\theta) = \theta_X(\iota_1, \ldots, \iota_m). \tag{1.2}$$

Proposition 1.1. *$Nat_{r_1,\ldots,r_m;t} \cong \pi_t(S_\ell(r_1) \otimes_\ell \cdots \otimes_\ell S_\ell(r_m))$.*

Proof. Since we have

$$\pi_{r_1} \times \cdots \times \pi_{r_m} \cong [S_\ell(r_1) \otimes_\ell \cdots \otimes_\ell S_\ell(r_m), -]_{\mathrm{Ho}(s\mathcal{A}_\ell)}$$

the result follows from Yoneda's lemma [21]. □

Note:

(1) There is an obvious map

$$\mathrm{Nat}_{r_1,\ldots,r_m;t} \times \mathrm{Nat}_{t;q} \to \mathrm{Nat}_{r_1,\ldots,r_m;q}$$

induced by composition.

(2) Nat is naturally an ℓ-module and f is naturally a linear map.

We now can address the issue of understanding possible relations among multioperations.

Corollary 1.2. *For $\theta \in Nat_{r_1,\ldots,r_m;t}$ then any expression for θ in $Nat_{r_1,\ldots,r_m;t}$ as a linear combination is formed and determined by a corresponding expression for $f(\theta)$ in $\pi_t(S_\ell(r_1) \otimes_\ell \cdots \otimes_\ell S_\ell(r_m))$. Furthermore, if $\psi \in Nat_{t;q}$ then $f(\psi \circ \theta) = f(\psi) \circ f(\theta)$, as composites of their homotopy representatives, in $\pi_q(S_\ell(r_1) \otimes_\ell \cdots \otimes_\ell S_\ell(r_m))$.*

Proof. This again follows from Yoneda's lemma [21]. □

Now, we are in a position to determine the full natural primary structure for homotopy in $s\mathcal{A}_\ell$. First, recall that for any field $\mathbb{F}$ we have

$$S_{\mathbb{F}}(V \oplus W) \cong S_{\mathbb{F}}(V) \otimes S_{\mathbb{F}}(W). \tag{1.3}$$

Next, we seek a natural map of ℓ-algebras

$$\phi_V : S_\ell(V \otimes_{\mathbb{F}_2} \ell) \to S_{\mathbb{F}_2}(V) \otimes_{\mathbb{F}_2} \ell$$

where V is a $\mathbb{F}_2$-module. This can be defined as the adjunction of the inclusion $V \otimes_k \ell \to I(S_{\mathbb{F}_2}(V) \otimes_{\mathbb{F}_2} \ell)$ (here $I : \mathcal{A}_\ell \to \mathcal{V}_\ell$ is the augmentation ideal functor).

Proposition 1.3. *The natural map $\phi : S_\ell((-) \otimes_{\mathbb{F}_2} \ell) \to S_{\mathbb{F}_2}(-) \otimes_{\mathbb{F}_2} \ell$ is an isomorphism of functors from $\mathbb{F}_2$-modules to $\mathcal{A}_\ell$.*

Proof. By the identity (1.3) and naturality, it is enough to provide a proof for one dimensional V, i.e. for $V \cong \mathbb{F}_2\langle x \rangle$. Then $\phi_V : \ell[x] \to \mathbb{F}_2[x] \otimes_{\mathbb{F}_2} \ell$ is determined algebraically by the value $\phi_V(x) = x \otimes_{\mathbb{F}_2} 1$. This is clearly an isomorphism.

Alternately, the inclusion $V \to IS_\ell(V \otimes_{\mathbb{F}_2} \ell)$ of $\mathbb{F}_2$-modules induces a natural $\mathcal{A}_\ell$-map $\zeta_V : S_{\mathbb{F}_2}(V) \otimes_{\mathbb{F}_2} \ell \to S_\ell(V \otimes_{\mathbb{F}_2} \ell)$ which serves as an inverse to ϕ_V. □

Corollary 1.4. *For $V \in s\mathcal{V}_{\mathbb{F}_2}$ there is a natural isomorphism*

$$\pi_*(S_\ell(V \otimes_{\mathbb{F}_2} \ell)) \cong \pi_*(S_{\mathbb{F}_2}(V)) \otimes_{\mathbb{F}_2} \ell.$$

As a consequence all natural primary homotopy operations for simplicial augmented ℓ-algebras and their relations are determined by $\pi_* S_{\mathbb{F}_2}(n)$ for all $n \in \mathbf{N}$.

Proof. The first statement follows from Proposition 1.3 and the faithful flatness of $(-) \otimes_{\mathbb{F}_2} \ell$. The second statement follows additionally from Corollary 1.2 and the Kunneth theorem. Recall that $S_\ell(n) \cong S_\ell(K_\ell(n))$ and we can take $K_\ell(n) = \ell\langle S^n \rangle \cong \mathbb{F}_2\langle S^n \rangle \otimes_{\mathbb{F}_2} \ell$, where S^n is a choice of simplicial set model for the n-sphere. □

Remarks. As a consequence of this corollary, we can conclude:

(1) An $\mathbb{F}_2$-basis for the natural operations of the homotopy of simplicial commutative $\mathbb{F}_2$-algebras will give an ℓ-basis for the natural operations of the homotopy of simplicial commutative ℓ-algebras for any field extension ℓ of $\mathbb{F}_2$.

(2) In order to give a complete account of the relations for the natural operations of the homotopy of a simplicial commutative ℓ-algebra, it is enough to combine the same account for simplicial commutative $\mathbb{F}_2$-algebras with the account that occurs for the same operations as they act upon ℓ, viewed as a constant simplicial commutative $\mathbb{F}_2$-algebra. This is where the Frobenius acting upon ℓ gets incorporated.

1.2. Homotopy operations at the prime 2

Let A be a simplicial commutative algebra of characteristic 2 (and, therefore, a simplicial $\mathbb{F}_2$-algebra). Associated to A is a chain complex, $(C(A), \partial)$, where, for each $n \in \mathbb{N}$, we have

$$C(A)_n = A_n, \quad \partial = \Sigma_{i=0}^n (-1)^i d_i = \Sigma_{i=0}^n d_i : C(A)_n \to C(A)_{n-1}.$$

It is standard that we have the identity [22]

$$\pi_n A \cong H_n(C(A)).$$

In [15], W. Dwyer showed the existence of natural chain maps

$$\Delta^k : (C(V) \otimes C(W))_{i+k} \to C(V \otimes W)_i \quad 0 \le k \le i,$$

where V and W are simplicial $\mathbb{F}_2$-modules, having the following properties:

(1) $\Delta^0 + T\Delta^0 T = \Delta + \phi_0$;
(2) $\Delta^k + T\Delta^k T = \partial\Delta^{k-1} + \Delta^{k-1}\partial$.

Here $T : C(V) \otimes C(W) \to C(W) \otimes C(V)$ is the twist map, $\Delta : C(V) \otimes C(W) \to C(V \otimes W)$ is the shuffle map [22, p. 243], and $\phi_k : C(V) \otimes C(W) \to C(V \otimes W)$ is the degree $(-k)$ map defined by

$$\phi_k(v \otimes w) = \begin{cases} 0 & \deg v \neq k \text{ or } \deg w \neq k; \\ v \otimes w & \text{otherwise.} \end{cases}$$

Note: Tensor product of chain complexes is graded tensor product and tensor product of simplicial modules is levelwise tensor product.

Now, for $x \in C(A)_n$ and $1 \le i \le n$, define $\Theta_i(x) \in C(A)_{n+i}$ by $\Theta_i(x) = \alpha_{n-i}(x)$ where

$$\alpha_t(x) = \mu\Delta^t(x \otimes x) + \mu\Delta^{t-1}(x \otimes \partial x),$$

and

$$\alpha_0(x) = \mu\Delta^0(x \otimes x),$$

where μ is the map $C(A \otimes A) \to C(A)$ induced by the product on A. As shown in [14, Section 3], these natural maps have the following properties:

(1) $\partial\Theta_i(x) = \Theta_i(\partial x)$ for $2 \le i \le n$;
(2) $\partial\Theta_n(x) = \mu\Delta(x \otimes \partial x)$;
(3) $\partial\Theta_1(x) = \Theta_1(\partial x) + x^2$;
(4) $\Theta_i(x+y) = \Theta_i(x) + \Theta_i(x) + \begin{cases} \partial\mu\Delta^{n-i-1}(x\otimes y) & 2 \le i < n; \\ \mu\Delta(x\otimes y) & i = n. \end{cases}$

From these chain properties for the Θ_i, there are induced homotopy operations

$$\delta_i : \pi_n A \to \pi_{n+i}A \quad 2 \le i \le n,$$

or, upon letting $\alpha_t = \delta_{n-t}$, we have

$$\alpha_t : \pi_n A \to \pi_{2n-t}A \quad 0 \le t \le n-2.$$

Note, in particular, that

$$\vartheta = \alpha_1. \tag{1.4}$$

The following is proved in [15,23]:

Theorem 1.5. *The homotopy operations δ_i have the following properties:*

(1) *δ_i is a homomorphism for $2 \le i \le n-1$ and $\delta_n = \gamma_2$—the divided square;*
(2) *δ_i acts on products as follows:*

$$\delta_i(xy) = \begin{cases} \delta_i(x)y^2 & \deg y = 0; \\ x^2\delta_i(y) & \deg x = 0; \\ 0 & \textit{otherwise}; \end{cases}$$

(3) *if $i < 2j$, then*

$$\delta_i\delta_j = \sum_{\frac{i+1}{2} \le k \le \frac{i+j}{3}} \binom{j-i+k-1}{j-s} \delta_{i+j-k}\delta_k.$$

Corollary 1.6. *The homotopy operations α_t have the following properties:*

(1) *α_t is a homomorphism for $1 \le i \le n-2$ and $\alpha_0 = \gamma_2$—the divided square;*
(2) *α_t acts on products as follows:*

$$\alpha_t(xy) = \begin{cases} \alpha_t(x)y^2 & \deg y = 0; \\ x^2\alpha_t(y) & \deg x = 0; \\ 0 & \textit{otherwise}; \end{cases}$$

(3) *if $s > t$, then*

$$\alpha_s\alpha_t = \sum_{\frac{s+2t}{3} \le q \le \frac{s+t-1}{2}} \binom{s-q-1}{q-t} \alpha_{s+2t-2q}\alpha_q.$$

Proof of Corollary 1.6. The first two items follow immediately from Theorem 1.5 using the identity $\alpha_t(x) = \delta_{n-t}(x)$ where $\deg x = n$. The last relation follows from (3) of Theorem 1.5 upon letting $j = n-t$, $i = 2n-s-t$, and $k = n-q$. $\square$

Let ℓ be a general field of characteristic 2 and let $\sigma : \ell \to \ell$ be the action of the *Frobenius*: $x^\sigma = x^2$. Our goal at present is to describe homotopy operations for simplicial commutative ℓ-algebras. Specifically, we will prove:

Theorem 1.7. *Let A be a simplicial augmented ℓ-algebra with $\mathrm{char}(\ell) = 2$. Then, for $2 \le i \le n$, the natural operation $\delta_i : \pi_n A \to \pi_{n+i}A$ satisfies properties (1)–(3) of Theorem 1.5. In particular, for $a, b \in \ell$ and $x, y \in \pi_n A$ we have*

$$\delta_i(ax+by) = a^\sigma\delta_i(x) + b^\sigma\delta_i(y) + \begin{cases} (ab)(xy) & i = n; \\ 0 & \textit{otherwise} \end{cases}$$

and for $u, v \in \pi_* A$

$$\delta_i((au)(bv)) = \begin{cases} (ab)^\sigma(\delta_i(u)v^2) & \deg v = 0; \\ (ab)^\sigma(u^2\delta_i(v)) & \deg u = 0; \\ 0 & \textit{otherwise}. \end{cases}$$

Furthermore, homotopy operations $\pi_n A \to \pi_{n+k}A$, as functors of simplicial augmented ℓ-algebras, are determined algebraically over ℓ by the operations $\delta_{i_1}\delta_{i_2}\dots\delta_{i_r}$ with $(i_1,\dots,i_r)$ an admissible sequence of degree k and excess $\le n$.

Recall that the *degree* of $I = (i_1, \ldots, i_r)$ is $i_1 + \cdots + i_r$ and the *excess* of I is $i_1 - i_2 - \cdots - i_r$. We will write throughout $\delta_I = \delta_{i_1}\delta_{i_2} \ldots \delta_{i_r}$. Finally, we call I *admissible* provided $i_{q-1} \geq 2i_q$ for all $2 \leq q \leq r$.

To prove Theorem 1.7, we need two lemmas. First, we record the following. See [14, 12.4.2].

Lemma 1.8. *Let A and B be simplicial commutative $\mathbb{F}_2$-algebras. Then the induced action of δ_i on $\pi_*(A) \otimes_{\mathbb{F}_2} \pi_*B$ is determined by*

$$\delta_i(x \otimes y) = \begin{cases} \delta_i(x) \otimes y^2 & \deg y = 0; \\ x^2 \otimes \delta_i(y) & \deg x = 0; \\ 0 & \textit{otherwise}. \end{cases}$$

Next, we define a *Γ_ℓ-algebra*, ℓ a field of characteristic 2, to be a graded commutative ℓ-algebra together with a set map $\gamma_2 : A_n \to A_{2n}$, $n \geq 2$, satisfying:

(1) $x^2 = 0$ for $x \in A_{\geq 1}$
(2) $\gamma_2(ax) = a^\sigma \gamma_2(x)$ for $a \in \ell$
(3) $\gamma_2(x + y) = \gamma_2(x) + \gamma_2(y) + xy$
(4) $\gamma_2(xy) = 0$ for $x, y \in A_{\geq 1}$
(5) $\gamma_2(xy) = x^2\gamma_2(y)$ for $x \in A_0$ and $y \in A_{\geq 2}$.

Given an ℓ-module V, we denote the *free Γ_ℓ-algebra* on V by $\Gamma_\ell[V]$.

Lemma 1.9. *For $n \geq 1$, we have*

$$\pi_* S_\ell(n) \cong \Gamma_\ell[\delta_I(\iota_n) | \text{ excess } (I) < n]$$

as Γ_ℓ-algebras.

Proof. For the case $\ell = \mathbb{F}_2$, see [15, Remark 2.3]. By Proposition 1.3, $\pi_* S_\ell(n) \cong (\pi_* S_{\mathbb{F}_2}(n)) \otimes_{\mathbb{F}_2} \ell$. The general result follows from Lemma 1.8 (with ℓ viewed as a constant simplicial $\mathbb{F}_2$-algebra) and the case $\ell = \mathbb{F}_2$. $\square$

Proof of Theorem 1.7. Since ℓ has characteristic 2, the operations δ_i are defined on $\pi_n A$ and satisfy (1) through (3) of Theorem 1.5. In particular, to compute $\delta_i(ax + by)$ it is enough, by Corollary 1.4, to compute

$$\delta_i(a\iota_n \otimes_\ell 1 + 1 \otimes_\ell b\iota_n) \in \pi_*(S_\ell(n)) \otimes_\ell \pi_*(S_\ell(n)).$$

Under the isomorphism (using Proposition 1.3 and Kunneth Theorem)

$$\pi_*(S_\ell(n)) \otimes_\ell \pi_*(S_\ell(n)) \cong (\pi_*(S_{\mathbb{F}_2}(n)) \otimes_{\mathbb{F}_2} \pi_*(S_{\mathbb{F}_2}(n))) \otimes_{\mathbb{F}_2} \ell,$$

$\delta_i(a\iota_n \otimes_\ell 1 + 1 \otimes_\ell b\iota_n)$ corresponds to $\delta_i((\iota_n \otimes_{\mathbb{F}_2} 1) \otimes_{\mathbb{F}_2} a + (1 \otimes_{\mathbb{F}_2} \iota_n) \otimes_{\mathbb{F}_2} b)$. Thus the desired result follows from Lemma 1.8. Similarly, to compute $\delta_i((au)(bv))$ it is enough to compute $\delta_i((a\iota_m) \otimes_\ell (b\iota_n)) \in \pi_*(S_\ell(m)) \otimes_\ell \pi_*(S_\ell(n))$, or, equivalently, $\delta_i((\iota_m \otimes_{\mathbb{F}2} \iota_n) \otimes_{\mathbb{F}_2} (ab)) \in (\pi_*(S_{\mathbb{F}_2}(m)) \otimes_{\mathbb{F}_2} \pi_*(S_{\mathbb{F}_2}(n))) \otimes_{\mathbb{F}_2} \ell$. This again can be computed using Lemma 1.8.

Finally, the last statement follows from Corollary 1.4 and Lemma 1.9. $\square$

Note: Theorem 1.7 shows that the operations δ_i and the relations (1)–(3) of Theorem 1.5 completely determine the homotopy operations for simplicial augmented algebras over general fields of characteristic 2. Thus the Galois group of ℓ over $\mathbb{F}_2$ produces no new homotopy operation of positive degree or alters the relations between them. This should not be surprising as the same considerations is known to hold rationally. See [18, Section 4].

1.3. Quillen's spectral sequence

We now focus on the André–Quillen homology for simplicial commutative ℓ-algebras. The key computational device relating André–Quillen homology and homotopy is Quillen's fundamental spectral sequence [2,24]. The main reference for computing with this spectral sequence over $\mathbb{F}_2$ is [14, Section 6] and we will spend this section reviewing and extending those results to a general field ℓ of characteristic 2.

To begin, we need to be more explicit about the functors $S_\ell(-)$. Let V be an ℓ-module. For $n \in \mathbb{N}$, define $S_{\ell,0}(V) = \ell$ and

$$S_{\ell,n}(V) = \ell \left\langle v_1 v_2 \ldots v_n \middle| \, v_i \in V \right\rangle.$$

Then

$$S_\ell(V) \cong \bigoplus_{n \in \mathbb{N}} S_{\ell,n}(V).$$

Next, let W be a non-negatively graded ℓ-module and define

$$\mathcal{S}_\ell(W) = \Gamma_\ell[\delta_I(w) \mid w \in W, \; I \text{ admissible}, \; \text{excess}(I) < \deg w] \tag{1.5}$$

which, by Corollary 1.6, can be expressed as

$$\mathcal{S}_\ell(W) \cong \Gamma_\ell[\alpha_1^{i_1}\alpha_2^{i_2}\dots\alpha_{n-2}^{i_{n-2}}(w) \mid w \in W,\ n = \deg w,\ i_1,\dots,i_{n-2} \in \mathbb{Z}_+]. \tag{1.6}$$

For $u \in \mathcal{S}_\ell(W)$, we define the *weight* of u, $\mathrm{wt}(u)$, as follows:

$$\mathrm{wt}(u) = \begin{cases} 0 & \text{if } u \in \ell; \\ 1 & \text{if } u \in W; \\ \mathrm{wt}(x) + \mathrm{wt}(y) & \text{if } u = xy; \\ 2\,\mathrm{wt}(x) & \text{if } u = \delta_i(x). \end{cases}$$

We then define, for $n \in \mathbb{N}$,

$$\mathcal{S}_{\ell,n}(W) = \ell\,\langle\, u \in \mathcal{S}_\ell(W) \mid\ \mathrm{wt}(u) = n\,\rangle\,.$$

Proposition 1.10. *For a simplicial $\mathbb{F}_2$-module V and $n \in \mathbb{N}$ there are a natural isomorphisms*

$$S_{\ell,n}(V \otimes_{\mathbb{F}_2} \ell) \cong S_{\mathbb{F}_2,n}(V) \otimes_{\mathbb{F}_2} \ell$$

and

$$\mathcal{S}_{\ell,n}((\pi_* V) \otimes_{\mathbb{F}_2} \ell) \cong \mathcal{S}_{\mathbb{F}_2,n}(\pi_* V) \otimes_{\mathbb{F}_2} \ell.$$

As a consequence, if W is a simplicial ℓ-module then

$$\pi_* S_{\ell,n}(W) \cong \mathcal{S}_{\ell,n}(\pi_* W).$$

Proof. The first two statements can be proved just as for Proposition 1.3. For the last statement, note that [14, Section 3] shows that the isomorphism holds when $\ell = \mathbb{F}_2$. Note also that a standard argument (e.g. via Postnikov towers) shows that there is a simplicial set X and a homotopy equivalence $W \simeq \ell\,\langle X\,\rangle$. Thus

$$\pi_* S_{\ell,n}(W) \cong \mathcal{S}_{\mathbb{F}_2,n}(\pi_* V) \otimes_{\mathbb{F}_2} \ell$$

where $V = \mathbb{F}_2\,\langle X\,\rangle$. Since $\pi_* W \cong (\pi_* V) \otimes_{\mathbb{F}_2} \ell$ it follows that

$$\mathcal{S}_{\ell,n}(\pi_* W) \cong \mathcal{S}_{\mathbb{F}_2,n}(\pi_* V) \otimes_{\mathbb{F}_2} \ell. \quad \square$$

We now follow [14, Section 6]. Let A be a simplicial augmented ℓ-algebra and let IA be its augmentation ideal. We may assume, using the standard model category structure [3, Section II.3], that A is almost free, i.e. $A_t \cong S_\ell(V_t)$ for all $t \geq 1$. Furthermore, the composite $V_t \subseteq IA_t \to QA_t$ to the indecomposables module is an isomorphism. We now form a decreasing filtration of A:

$$F_s = (IA)^s.$$

For A almost free,

$$E_s^0 A = F_s/F_{s+1} = (IA)^s/(IA)^{s+1} \cong S_{\ell,s}(QA).$$

Applying homotopy gives a spectral sequence

$$E^1_{s,t}A = \pi_t E_t^0 A \cong \pi_t S_{\ell,s}(QA) \Longrightarrow \pi_t A \tag{1.7}$$

with differentials

$$d_r : E^r_{s,t}A \to E^r_{s+r,t-1}. \tag{1.8}$$

This is called *Quillen's spectral sequence*.

Theorem 1.11. *For a simplicial augmented ℓ-algebra A there is a spectral sequence of algebras*

$$E^1 A_{s,t} = \mathcal{S}_{\ell,s}(H_*^Q(A))_t \Longrightarrow \pi_t A$$

with the following properties:

(1) *The spectral sequence converges if $\pi_0 A \cong \ell$. In particular, $E^r_{s,t}A = 0$ for $t < s$ for all $r \geq 1$.*
(2) *For $1 \leq r \leq \infty$ there are operations*

$$\delta_i : E^r_{s,t}A \to E^r_{2s,t+i}A \quad 2 \leq i \leq t$$

of indeterminacy $2r - 1$ with the following properties:
(a) *If $r = 1$, then δ_i coincides with the induced operation $\mathcal{S}_{\ell,s}(H_*^Q(A))_t \to \mathcal{S}_{\ell,2s}(H_*^Q(A))_{t+i}$.*

(b) *If $x \in E^rA$ and $2 \le i < t$ then $\delta_i(x)$ survives to $E^{2r}A$ and*

$$d_{2r}\delta_i(x) = \delta_i(d_r x)$$

$$d_r\delta_t(x) = x d_r x$$

modulo indeterminacy.

(c) *The operations on E^rA are induced by the operations on $E^{r-1}A$ and the operations on $E^\infty A$ are induced by the operations on E^rA for $r < \infty$.*

(d) *The operations on $E^\infty A$ are induced by the operations on π_*A.*

(e) *Up to indeterminacy, the operations on E^rA satisfy the properties of Theorem 1.7.*

Before we indicate a proof of this omnibus result, a word of explanation is needed. First, an element $y \in E^r_{s,t}A$ is said to be defined up to *indeterminacy q* provided y is a coset representitive for a particular element of $E^r_{s,t}A/B^q_{s,t}A$ where

$$B^q_{s,t}A \subseteq E^r_{s,t}A \quad q \ge r$$

is the ℓ-module of elements of $E^r_{s,t}A$ which survive to $E^q_{s,t}A$ but have zero residue class.

Also, if A is almost free, and hence cofibrant as a simplicial augmented ℓ-algebra, then

$$\pi_*(QA) \cong H^Q_*(A).$$

Cf. [8, Section 1].

Proof. First, if A is almost free, we have a pairing

$$\pi_t(S_{\ell,s}(QA)) \otimes \pi_{t'}(S_{\ell,s'}(QA)) \stackrel{(\mu\Delta)_*}{\to} \pi_{t+t'}(S_{\ell,s+s'}(QA))$$

which gives a pairing

$$E^1_{s,t}A \otimes E^1_{s',t'}A \to E^1_{s+s',t+t'}A$$

and induces an algebra structure on the spectral sequence.

For (1), we simply note that if A is connected then $\mathcal{S}_{\ell,s}(H^Q_*(A)) = 0$ for $t > s$. Convergence now follows from standard convergence theorems. Cf. [22].

For (2), we have a commutative diagram

$$\begin{array}{ccccc} C((IA)^s) \otimes_{\mathbb{F}_2} C((IA)^s) & \xrightarrow{\bar{\alpha}_t} & C((IA)^s \otimes_{\mathbb{F}_2} (IA)^s) & \longrightarrow & C((IA)^s \otimes_\ell (IA)^s) \\ \sigma \uparrow & & & & \downarrow \mu \\ C((IA)^s) & \xrightarrow{\alpha_t} & C((IA)^{2s}) & = & C((IA)^{2s}) \end{array}$$

where $\sigma(u) = u \otimes u$ and $\bar{\alpha}_t(a \otimes b) = \Delta^t(a \otimes b) + \Delta^{t-1}(a \otimes \partial b)$. This induces a map

$$\Theta_i : (IA)^s \to (IA)^{2s},$$

by again setting $\Theta_i(u) = \alpha_{n-i}(u)$ where $n = \deg u$.

Let $x \in E^r_{s,t}A$. Then, modulo $(IA)^{s+1}$, x is represented by $u \in (IA)^s$ with the property that $\partial u \in (IA)^{s+r}$. The class of u is not unique, but may be altered by adding elements $\partial b \in (IA)^s$ with $b \in (IA)^{s-r+1}$.

Define $\delta_i(x) \in E^r_{2s,t+i}A$ to be the residue class of $\Theta_i(u)$. Since

$$\partial\Theta_i(u) = \Theta_i(\partial u) \in (IA)^{2s+2r} \quad 2 \le i < t,$$

and

$$\partial\Theta_t(u) = \mu\Delta(u \otimes \partial u) \in (IA)^{2s+r}.$$

Thus $\delta_i(x)$ is defined in $E^r_{2s,t+i}A$ and survives to $E^{2r}A$ with $d_{2r}\delta_i(x) = \delta_i(d_r x)$ for $2 \le i < t$. Also $d_r\delta_t(x) = x d_r x$. This gives us (b).

Now we have a commuting diagram

$$\begin{array}{ccc} \pi_t((IA)^s) & \xrightarrow{(\Theta_i)_*} & \pi_{t+i}((IA)^{2s}) \\ \downarrow & & \downarrow \\ \pi_t A & \xrightarrow{\delta_i} & \pi_{t+i}A \end{array}$$

and an induced diagram

$$\begin{array}{ccc} \pi_t((IA)^s) & \xrightarrow{(\Theta_i)_*} & \pi_{t+i}((IA)^{2s}) \\ \downarrow & & \downarrow \\ \pi_t((IA)^s/(IA)^{s+1}) & \longrightarrow & \pi_{t+i}((IA)^{2s}/(IA)^{2s+1}) \\ \downarrow\cong & & \cong\downarrow \\ \mathcal{S}_{\ell,s}(H^Q_*(A))_t & \xrightarrow{\delta_i} & \mathcal{S}_{\ell,2s}(H^Q_*(A))_{t+i} \end{array}$$

It is now straightforward to check (a), (c), (d), and (e). $\square$

2. Proof of Theorem C

The goal of this section will be to provide a proof of the Theorem C. This will involve a careful study of a certain map, the character map, defined on the homotopy of of simplicial augmented algebras with finite André–Quillen homology, whose non-triviality will give us Theorem C. In fact, in the process of analysing this character map, we will be able to establish an upper bound on the top non-trivial degree of the André–Quillen homology in terms of the non-nilpotence of certain operations acting on homotopy.

2.1. Nilpotency in homotopy

For fields of characteristic 2, the computation of $\pi_* S_{\mathbb{F}_2}$ can be traced back to [15]. In particular, we will be interested in two particular operations. First, we will note that for $A \in s\mathcal{A}_\ell$, $\pi_* A$ is naturally a divided power algebra [12,14]. Therefore, there is a *divided square*

$$\gamma_2 : \pi_n A \to \pi_{2n} A.$$

There is also an operation

$$\vartheta : \pi_n A \to \pi_{2n-1} A$$

which we call the *André operation* because of the role it played in [6]. In that paper, it is first demonstrated that Gulliksen's result [5] showing that the deviations and simplicial dimensions coincide for rational local rings, a result that establishes Quillen's conjecture for rational local rings, cannot be extended to prime characteristic settings. In the notion of [15],

$$\vartheta = \delta_{n-1}. \tag{2.9}$$

A useful basic relation between the two operations is

$$\vartheta \gamma_2 = 0. \tag{2.10}$$

Lemma 2.1. *Let $W \in s\mathcal{V}_\ell$, with $\mathrm{char}(\ell) = 2$, and let $n \in \mathbb{N}$ be so that $\pi_j W \neq 0$ implies $n \geq j \geq 1$. Then*

(1) *$\gamma_2 = 0$ on $\pi_* S_\ell(W)$ provided $n = 1$;*
(2) *$\vartheta = 0$ on $\pi_* S_\ell(W)$ provided $n = 2$.*

Proof. By Corollary 1.4, it is enough to provide a proof for $\ell = \mathbb{F}_2$. For $n = 1$, $\pi_* S_\ell(W)$ is a free exterior algebra generated by $\pi_1 W$, which has trivial γ_2-action. For $n = 2$, $\pi_* S_\ell(W)$ is a free divided power algebra generated by $\pi_* W$. Cf. [25]. Thus $\pi_* S_\ell(W)$ has trivial ϑ-action by relation (2.10). □

Given $A \in s\mathcal{A}_\ell$ with $\mathrm{char}(\ell) = 2$, we define A to be

(1) γ_2-**nilpotent** provided that for each $x \in \pi_* A$ there is an $s > 0$ such that $\gamma_2^s(x) = 0$ for and
(2) **André nilpotent** provided that for each $x \in \pi_* A$ there is an $s > 0$ such that $\vartheta^s(x) = 0$.

Next, let R be a Noetherian ring and a let A be a simplicial commutative R-algebra with Noetherian homotopy such that $R \to \pi_0 A$ is a surjection. For $\mathfrak{p} \in \mathrm{Spec}(\pi_0 A)$ with $\mathrm{char}(k(\mathfrak{p})) = 2$, we call A

(1) γ_2-**nilpotent at** $\mathfrak{p}$ provided $A(\mathfrak{p})$ is γ_2-nilpotent over $k(\mathfrak{p})$, and
(2) **André nilpotent at** $\mathfrak{p}$ provided $A(\mathfrak{p})$ is André nilpotent over $k(\mathfrak{p})$.

Proposition 2.2. *Let A be a simplicial commutative R-algebra with Noetherian homotopy and $\mathfrak{p} \in \mathrm{Spec}(\pi_0 A)$ such that $\mathrm{char}(k(\mathfrak{p})) = 2$. Then*

(1) *A is γ_2-nilpotent at $\mathfrak{p}$ provided A is a homotopy 1-intersection at $\mathfrak{p}$;*
(2) *A is André-nilpotent at $\mathfrak{p}$ provided A is a homotopy 2-intersection at $\mathfrak{p}$.*

Proof. Both follow from the definitions and Lemma 2.1. □

2.2. Connected envelopes and the character map

We begin by providing a strategy for proving Theorem C. This will first involve reviewing the concept of connected envelopes from [8]. We then construct the notion of a character map for connected simplicial augmented algebras with finite André–Quillen homology and state a conjecture regarding this map whose validity implies Theorem C.

Given A in $s\mathcal{A}_\ell$, which is connected, we define its **connected envelopes** to be a sequence of cofibrations

$$A = A(1) \xrightarrow{j_1} A(2) \xrightarrow{j_2} \cdots \xrightarrow{j_{n-1}} A(n) \xrightarrow{j_n} \cdots$$

with the following properties:

(1) For each $n \geq 1$, $A(n)$ is a $(n-1)$-connected.
(2) For $s \geq n$,

$$H_s^Q A(n) \cong H_s^Q A.$$

(3) There is a cofibration sequence

$$S_\ell(H_n^Q A, n) \xrightarrow{f_n} A(n) \xrightarrow{j_n} A(n+1).$$

Here we write, for $B \in s\mathcal{A}_\ell$, $H_s^Q(A) := D_s(A|\ell;\ell)$ and, for $V \in \mathcal{V}_\ell$, $S_\ell(V, m) := S_\ell(K(V, m))$. Existence of connected envelopes is proved in [8, Section 2].

Note: Paul Goerss has pointed out that connected envelopes can also be constructed through a "reverse" decomposition via collapsing skeleta on the canonical CW approximation.

Now, for $A \in s\mathcal{A}_\ell$ connected, define the *André–Quillen dimension* of A to be

$$\text{AQ-dim}(A) = \max\{m \in \mathbb{N} \mid H_m^Q(A) \neq 0\}.$$

Assume that $n = \text{AQ-dim}\, A < \infty$. Then

$$A(n) \simeq S_\ell(H_n^Q(A), n).$$

Cf. [8, (2.1.3)]. Summarizing, we have

Proposition 2.3. *For $A \in s\mathcal{A}_\ell$ connected and $\text{AQ-dim}\, A < \infty$ there is a natural map*

$$\phi_A : A \to S_\ell(H_n^Q(A), n),$$

where $n = \text{AQ-dim}\, A$, with the property that $H_n^Q(\phi_A)$ is an isomorphism.

Now, assuming $\text{char}(\ell) > 0$, we noted that $\pi_* B$ is naturally a divided power algebra. Given a divided power algebra Λ in characteristic p, let $J \subset \Lambda$ be the divided power ideal generated by all decomposables $w_1 w_2 \ldots w_r$ and $\gamma_p(z)$ with $w_1, w_2, \ldots, w_r, z \in \Lambda_{\geq 1}$. Define the *$\Gamma$-indecomposables* to be

$$Q_\Gamma \Lambda = \Lambda / J.$$

Given $A \in s\mathcal{A}_\ell$ connected and $n = \text{AQ-dim}\, A$ finite, we define the *character map* of A to be

$$\Phi_A = Q_\Gamma \pi_*(\phi_A) : Q_\Gamma \pi_* A \to Q_\Gamma \pi_*(S_\ell(H_n^Q(A), n)).$$

Now, for $B \in s\mathcal{A}_\ell$, the action of the André operation ϑ on $\pi_* B$ induces an action on $Q_\Gamma \pi_* B$ by the relation (2.10) and the fact that ϑ kills decomposables of elements of positive degree. Cf. [26, (8.9)].

Theorem 2.4. *Let $A \in s\mathcal{A}_\ell$ be connected with $\text{char}(\ell) = 2$ and $H_*^Q(A)$ a non-trivial finite graded ℓ-module. Then Φ_A is non-trivial.*

Proof of Theorem C. Let $n = \text{AQ-dim}\, B$ where $B = A \otimes_R^{\mathbf{L}} \ell$ with $\ell = k(\mathfrak{p})$. By Corollary 1.4 and [14, (3.5)], ϑ acts non-nilpotently on every non-trivial element of $Q_\Gamma \pi_*(S_\ell(H_n^Q(B), n))$ if $n \geq 3$. Therefore if $\pi_* B$ is André nilpotent then Theorem 2.4 implies that $n \leq 2$. Thus B is a homotopy 2-intersection by [8, (2.2)].

Since $H_*^Q(B) \cong H_*^Q(A(\mathfrak{p}))$, if A is additionally γ_2-nilpotent at $\mathfrak{p}$ then A is a homotopy 1-intersection at $\mathfrak{p}$, as $\pi_* A(\mathfrak{p})$ is free as a divided power algebra. $\square$

The goal of the next subsection will be to provide a proof of Theorem 2.4.

2.3. Non-triviality of the character map

We now proceed to prove Theorem 2.4. We will in fact prove a more general theorem. Specifically:

Theorem 2.5. *Let A be a simplicial augmented ℓ-algebra ($\text{char}(\ell) = 2$) such that $H_*^Q(A)$ is finite graded as an ℓ-module. Let $n = \text{AQ-dim}\, A$ and assume $n \geq 2$. Then there exists $x \in \pi_* A$ and $y \neq 0 \in H_n^Q(A)$ such that under the map*

$$\pi_* \phi_A : \pi_* A \to \pi_* S_\ell(H_n^Q(A), n)$$

we have

$$(\pi_* \phi_A)(x) = \alpha_{n-2}^t(y)$$

for some $t \geq 1$.

With this result, we can provide the following:

Proof of Theorem 2.4. Assume $n = \text{AQ-dim}\, A \geq 3$. Let $y \in H_n^Q(A)$, and $x \in \pi_* A$ satisfy the properties of Theorem 2.5. By Eq. (1.6), $\alpha_{n-2}^t(y) \neq 0$ in $Q_\Gamma \pi_*(S_\ell(H_n^Q(A), n))$ for all $t \geq 1$. We conclude that $\Phi_A(x) \neq 0$.

If $n \leq 2$, then Φ_A is a surjection and, hence, non-trivial. $\square$

Now, in order to prove Theorem 2.5 we will need to know something about the annihilation properties of homotopy operations. Specifically, we will focus on composite operations of the form

$$\theta(s,t) = \delta_{2^s}\delta_{2^{s-1}}\ldots\delta_{2^{t+1}} \quad s > t$$

(where we set $\theta(t+1,t) = \delta_{2^{t+1}}$).

Lemma 2.6. *Let $i \geq 2$ and $t \geq 1$ be such that $2^t < i$. Then $\theta(s,t)\delta_i = 0$ for $s \gg t$.*

Proof. Write $i = 2^{t-1} + n$ with $n \geq 1$. Note first that an application of the relation Theorem 1.5(3) shows that for any $t \geq 1$,

$$\delta_{2^{t+1}}\delta_{2^t+1} = \delta_{2^{t+1}}\delta_{2^t+2} = 0.$$

We thus assume, by induction, that for any t and $0 < j < n$, there exists $s \gg t$ such that

$$\theta(s,t)\delta_{2^t+j} = 0.$$

By another application of the relation Theorem 1.5 (3), we have

$$\delta_{2^{t+1}}\delta_{2^t+n} = \sum_{1\leq r\leq \frac{n}{3}} \binom{n+r-1}{n-r} \delta_{2^{t+1}+n-r}\delta_{2^t+r}.$$

Notice that, for each such r, $2^{t+1} < 2^{t+1} + n - r < 2^{t+1} + n$. Thus, by induction, we can find $s \gg t+1$ so that

$$\theta(s,t+1)\left(\sum_{1\leq r\leq \frac{n}{3}} \binom{n+r-1}{n-r} \delta_{2^{t+1}+n-r}\delta_{2^t+r}\right) = 0.$$

We conclude that

$$\theta(s,t)\delta_{2^t+n} = \theta(s,t+1)\delta_{2^{t+1}}\delta_{2^t+n} = 0. \quad \square$$

Corollary 2.7. *Let $I = (i_1, \ldots, i_k)$ be an admissible sequence and let $t < k$. Then $\theta(s,t)\delta_I = 0$ for $s \gg t$.*

Proof. Since I is admissible, then

$$i_1 \geq 2i_2 \geq \cdots \geq 2^{k-1} i_k \geq 2^k > 2^t.$$

Thus, by Lemma 2.6,

$$\theta(s,t)\delta_I = (\theta(s,t)\delta_{i_1})\delta_{i_2}\ldots\delta_{i_k} = 0$$

for $s \gg t$. $\square$

Proposition 2.8. *Let A be a connected simplicial augmented ℓ-algebra,* $\text{char}(\ell) = 2$. *In Quillen's spectral sequence for A, let $y \neq 0$ in $E^1_{1,n}A \cong H_n^Q(A)$, $n \geq 2$. Then there exists $s \geq 1$ such that $\alpha_{n-2}^s(y) \in E^1_{2^s, n+2^{s+1}-2}A$ survives to $E^\infty A$.*

Proof. Choose $m \geq 1$ and suppose $\alpha_{n-2}^m(y)$ survives to $E^r A$, $r \geq 1$. By Theorem 1.11 (2) (a), we may assume that $r \geq 2^m$. Let $w = d_r([\alpha_{n-2}^m(y)]) \in E^r_{2^m+r, n+2^{m+1}-3}A$ by (1.8). By Theorem 1.11 (1), $w = 0$ provided $n + 2^m - 2 \leq r$. Thus if $r \geq n + 2^m - 2$ then the class of $\alpha_{n-2}^m(y)$ survives to $E^\infty A$ as all subsequent differentials will satisfy the same criterion.

Suppose next that $r < n + 2^m - 2$. Write $n + 2^m - q = r$ with $n \geq q > 2$. Assume, by induction, that if for some m the class of $\alpha_{n-2}^m(y)$ survives to $E^{n+2^m-j}_{2^m, n+2^{m+1}-2}A$ for $q > j$ then there exists $s \gg m$ such that the class of $\alpha_{n-2}^s(y)$ survives to $E^\infty A$. Again, let $w = d_r([\alpha_{n-2}^m(y)]) \in E^r_{2^m+r, n+2^{m+1}-3}A$. Choose $u \in E^1_{2^m+r, n+2^{m+1}-3}A$ to represent the class w. By Theorem 1.11 and Proposition 1.3, we have

$$u = \begin{cases} \sum_{I,l} a_{I,l}\delta_I(x_l) + \sum_J b_J z_J & r = 2^k - 2^m,\ k > m; \\ \sum_J b_J z_J & \text{otherwise} \end{cases}$$

where $I = (i_1, \ldots, i_k)$ and $J = (j_1, \ldots, j_r)$ are sequences with I admissible, $a_{I,k}, b_J \in \ell$, and $z_J = z_{j_1} z_{j_2} \ldots z_{j_{r+2^m}}$ with $x_k, z_{j_1}, \ldots, z_{j_{r+2^m}} \in H_*^Q(A)$.

First assume that $r \neq 2^k - 2^m$. Then $d_r([\alpha^m_{n-2}(y)]) = [u] \in E^r_{2^m+r,n+2^{m+1}-3}A$ with $u \in E^1_{2^m+r,n+2^{m+1}-3}A$ decomposable. Note again that $\deg u > 2^m$. Thus, by Theorem 1.11 (2) (b), (c), and (e), $d_{2r}(\delta_{2^{m+1}}[\alpha^m_{n-2}(y)]) = \delta_{2^{m+1}} d_r([\alpha^m_{n-2}(y)]) = \delta_{2^{m+1}}[u] = 0$. Thus $[\alpha^{m+1}_{n-2}y]$ survives to $E^{2r+1}_{2^{m+1},n+2^{m+2}-2}A$. Now, let $2r+1 = n+2^{m+1}-j$ and recall that $r = n+2^m-q \geq 2^m$. Then

$$j = (n+2^{m+1}) - (2n+2^{m+1}-2q) - 1 = 2q-n-1 = q-(n-q)-1 < q.$$

Thus, by induction, there exists $s \gg m$ such that $[\alpha^s_{n-2}(y)]$ survives to E^∞.

Now assume that $r = 2^k - 2^m$ with $k > m$. By definitions of α_{n-2} and $\theta(m,t)$,

$$\alpha^m_{n-2}(y) = \theta(m,0)(y).$$

By Theorem 1.11 (2) (b) and (c), for $e > m$

$$d_{2^{e-m}r}([\theta(e,0)y]) = \theta(e,m)d_r([\theta(m,0)(y)]) = \theta(e,m)w.$$

By Theorem 1.11 (2) (c) and (e) and Theorem 1.7, $\theta(e,m)w$ is represented by

$$\sum_{I,l} a^{2^{e-m}}_{I,l}\theta(e,m)\delta_I(z_l) \quad \text{modulo indeterminacy.}$$

Note that $2^m < \deg u$ so we can assume there are no decomposables in our choice of representative for $\theta(e,m)w$. As indicated above, we have for each $I = (i_1,\ldots,i_k)$ that $k > m$. Thus, by Corollary 2.7, since the sum is finite, there exists $e \gg m$ such that

$$\theta(e,m)\delta_I = 0 \quad \text{for all } I.$$

Thus $d_{2^{e-m}r}([\theta(e,0)y]) = 0$ modulo indeterminacy. Therefore $[\theta(e,0)y]$ survives to $E^{2^{e-m}r+1}_{2^e,n+2^{e+1}-2}A$, so, by the previous case, there exists $s \gg e$ such that $[\alpha^s_{n-2}(y)] = [\theta(s,0)y]$ survives to E^∞. $\square$

Proof of Theorem 2.5. Choose $y \in H^Q_n(A) \cong E^1_{1,n}A$ and choose $s \geq 1$ such that $\alpha^s_{n-2}(y) \in E^1A$ survives to $E^\infty A$, which exists by Proposition 2.8. Under the induced map

$$E^r(\phi_A) : E^1A \to E^rS_\ell(H^Q_n(A), n)$$

we have

$$E^r(\phi_A)([\alpha^s_{n-2}(y)]) = [\alpha^s_{n-2}(y)]$$

for all $\infty \geq r \geq 1$. But, since

$$E^1S_\ell(H^Q_n(A), n) \cong E^\infty S_\ell(H^Q_n(A), n),$$

we can conclude that $E^\infty(\phi_A)([\alpha^s_{n-2}(y)]) \neq 0$. Thus we can find a nontrivial $x \in \pi_*A$ which is represented by $\alpha^s_{n-2}(y)$ in $E^\infty A$ such that $(\pi_*\phi_A)(x) = \alpha^s_{n-2}(y) \neq 0$. $\square$

3. Proof of Theorem E

In order to get a handle on the Nilpotence Conjecture, we will need more technical aspects of commutative algebra and a deeper analysis of the chains of a simplicial commutative algebra of characteristic 2.

In order to uncover nilpotence behavior in the homotopy of a simplicial commutative R-algebra, we will need to focus more carefully at local behavior and seek to strip out aspects that minimally contribute to the behavior of the cotangent complex.

Proposition 3.1. *Suppose A is a simplicial commutative R-algebra with $R \to \pi_0A$ a surjection and let $\mathfrak{p} \in \operatorname{Spec} R$ with residue field ℓ. Then there exists a local ring $(R', \mathfrak{m})$, a simplicial commutative R'-algebra A', and a homotopy commutative diagram*

$$\begin{array}{ccc} R & \xrightarrow{\eta} & A \\ \phi\downarrow & & \downarrow\psi \\ R' & \xrightarrow{\eta'} & A' \end{array} \tag{3.11}$$

with the following properties:

(1) *ϕ is a complete intersection homomorphism;*
(2) $\operatorname{depth}(\mathfrak{m}) = 0$*;*
(3) $D_*(A|R;\ell) \cong D_*(A'|R';\ell)$*;*
(4) *η' induces a surjection of local rings $\eta'_* : R' \to \pi_0A'$;*
(5) *If A has finite Noetherian homotopy then A' has finite Noetherian homotopy.*

Proof. Let $x_1, \ldots, x_r$ be a maximal R-subsequence of a minimal generating set for $\mathfrak{p}$. We define

$$R' = R_{\mathfrak{p}}/(x_1, \ldots, x_r).$$

Then $\mathfrak{m} = \mathfrak{q}/(x_1, \ldots, x_r)\mathfrak{q}$ has depth 0 since it contains only zero divisors. Furthermore, the composite $R \to R_{\mathfrak{p}} \to R'$ is a complete intersection homomorphism by definition. Cf. [4].

Now, let $A' = A \otimes^{\mathbf{L}}_R R'$. Then

$$D_*(A|R;\ell) \cong D_*(A \otimes^{\mathbf{L}}_R R'|R';\ell) = D_*(A'|R';\ell)$$

which follows from flat base change [13, (2.4)]. Applying π_0 to the map $R' \to A \otimes^{\mathbf{L}}_R R'$ gives the map $R' \cong R \otimes_R R' \to \pi_0(A) \otimes_R R'$ which is a surjection. Thus $R' \to \pi_0 A'$ is a surjection.

Finally, if A has finite Noetherian homotopy then, by [3, Section II.6], there is a Kunneth spectral sequence

$$E^2_{s,t} = \mathrm{Tor}^R_s(\pi_t A, R') \Longrightarrow \pi_{s+t}A'.$$

Since $R \to R'$ is a complete intersection homomorphism, $\mathrm{fd}_R R' < \infty$. Thus $\pi_* A'$ will be a finite module over $\pi_0 A' \cong (\pi_0 A) \otimes_R R'$. $\square$

We will call a diagram (3.11) satisfying the conditions (1)–(5) above a *local homotopy reduction* for A.

Next, let A be a simplicial commutative $\mathbb{F}_2$-algebra and let $(C(A), \partial)$ be the associated chain complex. The following is proved in [12,15].

Proposition 3.2. *The shuffle map $\Delta : C(A) \otimes_{\mathbb{F}_2} C(A) \to C(A \otimes_{\mathbb{F}_2} A)$ induces a divided power algebra structure on $C(A)$. Specifically, for each $k \in \mathbb{Z}_+$, there is a function $\gamma_k : C(A)_n \to C(A)_{kn}$ satisfying:*

(1) $\gamma_0(x) = 1$ *and* $\gamma_1(x) = x$
(2) $\gamma_h(x)\gamma_k(x) = \binom{h+k}{h}\gamma_{h+k}(x)$
(3) $\gamma_k(x+y) = \sum_{r+s=k}\gamma_r(x)\gamma_s(y)$
(4) $\gamma_k(xy) = 0$ *for* $k \geq 2$ *and* $x, y \in C(A)_{\geq 1}$
(5) $\gamma_k(xy) = x^k\gamma_k(y)$ *for* $x \in C(A)_0$ *and* $y \in C(A)_{\geq 2}$
(6) $\gamma_k(\gamma_2(x)) = \gamma_{2k}(x)$
(7) $\partial\gamma_k(x) = (\partial x)\gamma_{k-1}(x)$
(8) $u \in C(A)_n$ *a cycle then, for* $[u] \in \pi_n A$, $\delta_n([u]) = [\gamma_2(u)]$.

Let $A \to B$ be a map of simplicial commutative $\mathbb{F}_2$-algebras and $\rho : C(A) \to C(B)$ the induced map of chain complexes. Then for $u \in C(A)_n$ and all $n > i \geq 0$

$$\rho(\alpha_i(u)) = \alpha_i(\rho(u))$$

where $\alpha_i = \Theta_{n-i}$. Recall (1.4) that $\vartheta = \alpha_1$.

Lemma 3.3. *Let $A \to B$ be a map of simplicial commutative $\mathbb{F}_2$-algebras and suppose $\pi_s A = 0$ for $s \gg 0$. Let $u \in C(A)_n$, $n \geq 3$, such that $\rho(\partial u) = 0$. Then $\rho(u)$ is a cycle in $C(B)$ and $\vartheta^r([\rho(u)]) = 0$ in $\pi_*(B)$ for $r \gg 0$ provided $\gamma_2^r(\partial u) = 0$ in $C(A)$ for $r \gg 0$.*

Proof. First, in $C(A)$, we have, by an induction using the formulas for Θ_i from Section 2.2, that

$$\partial\vartheta^r(u) = \gamma_2^r(\partial u).$$

Since $\gamma_2^r(\partial u) = 0$ for $r \gg 0$ and $H_s(C(A)) = 0$ for $s \gg 0$, it follows that $\vartheta^r(u)$ is a boundary in $C(A)$ for $r \gg 0$. We conclude that $\vartheta^r([\rho(u)]) = [\rho(\vartheta^r(u))] = 0$ in $\pi_*(B)$. $\square$

Corollary 3.4. *Let $A \to B$ be a level-wise surjection of simplicial commutative $\mathbb{F}_2$-algebras such that γ_2 acts locally nilpotently on $(\partial C(A)) \cap \ker\rho$ and $\pi_s A = 0$ for $s \gg 0$. Then B is André nilpotent.*

Proof. Given $x \in \pi_n B$ with $n \geq 3$, let $w \in C(B)_n$ be a cycle representitive for x and choose $u \in C(A)$ such that $\rho(u) = w$. Then $\rho(\partial u) = 0$ and $\gamma_2^r(\partial u) = 0$ for $r \gg 0$ by assumption. Thus $\vartheta^r(x) = 0$ for $r \gg 0$ by Lemma 3.3. $\square$

Proof of Theorem E. Let A be a cofibrant simplicial commutative R-algebra with finite Noetherian homotopy, $R \to \pi_0 A$ a surjection, and R a Cohen–Macaulay ring of characteristic 2. Assume further that $D_s(A|R;-) = 0$ for $s \gg 0$, as a functor of $\pi_0 A$-modules. Note that A has an induced simplicial $\mathbb{F}_2$-algebra structure. Now, choose a $\mathfrak{p} \in \mathrm{Spec}(\pi_0 A)$ and a local homotopy reduction, $(R', \mathfrak{m}) \to A'$, of A at $\mathfrak{p}$, which exists by Proposition 3.1. We then have, by Proposition 3.1, [17], and [27, Section 5], that $(R', \mathfrak{m})$ is a Cohen–Macaulay ring of depth zero and, hence, locally Artin.

We will now show that A' is André nilpotent at $\mathfrak{p}$. Let $\ell = k(\mathfrak{p})$ (so that $\mathrm{char}(\ell) = 2$). Let $B = A' \otimes^{\mathbf{L}}_{R'} \ell = A' \otimes_{R'} \ell$. Then

$$C(B) \cong C(A') \otimes_{R'} \ell \cong C(A')/\mathfrak{m}C(A'). \tag{3.12}$$

Thus $\rho : C(A') \to C(B)$ is a surjection and $\ker \rho = \mathfrak{m}C(A')$. Since R is locally Artinian,

$$\mathfrak{m}^s = 0 \quad s \gg 0. \tag{3.13}$$

Cf. [27, 2.3]. Let $a, b \in \mathfrak{m}$ and let $x, y \in C(A')$ of degrees ≥ 2. By Proposition 3.2 (3) and a straightforward induction,

$$\gamma_2^r(ax + by) = a^{2^r}\gamma_2^r(x) + b^{2^r}\gamma_2^r(y) \quad \text{modulo decomposables}.$$

Thus, by (3.13) and Proposition 3.2 (4), $\gamma_2^r(ax + by) = 0$ for $r \gg 0$. Hence, by a further induction, γ_2 acts locally nilpotently on $\mathfrak{m}C(A')$. Therefore, by Corollary 3.4, B is André nilpotent. Finally,

$$B = A' \otimes^{\mathbf{L}}_{R'} \ell \cong (A \otimes_R R') \otimes_{R'} \ell \cong A \otimes_R \ell \cong A(\mathfrak{p})$$

so $A(\mathfrak{p})$ is André nilpotent. □

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