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CE EQUIVALENCE AND SHAPE EQUIVALENCE OF 1-DIMENSIONAL COMPACTA

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In this paper the relationship between CE equivalence and shape equivalence for locally connected, 1-dimensional compacta is investigated. Two theorems are proved. The first asserts that every path connected planar continuum is CE equivalent either to a bouquet of circles or to the Hawaiian earring. The second asserts that for every locally connected, 1-dimensional continuum X there is a cell-like map of X onto a planar continuum. It follows that CE equivalence and shape equivalence are the same for the class of all locally connected, 1-dimensional compacta. In addition, an example of Ferry is generalized to show that for every $n \geq 1$ there exists an n-dimensional, LC^{n-2} continuum Y such that $\mathrm{Sh}(Y) = \mathrm{Sh}(S^1)$ but Y is not CE equivalent to S^1.

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cell-like map | shape equivalence | compacta
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Introduction

Two finite dimensional compacta X and Y are said to be *CE equivalent* if there exists a third finite dimensional compactum Z and cell-like maps $f: Z \to X$ and $g: Z \to Y$. It is easily seen that this relationship induces an equivalence relation on the class of all finite dimensional compacta. In this paper we investigate the relationship between CE equivalence and shape equivalence in case the compacta X and Y are locally connected and 1-dimensional.

Ferry [4] has shown that homotopy equivalent compacta are always CE equivalent. In addition, every CE map of finite dimensional compacta is a shape equivalence [8]. Thus it was natural to ask whether CE equivalence and shape equivalence are the same for finite dimensional compacta, but Ferry [5] gave an example which shows that this is not the case. Let X denote the circle with a spiral attached as in Fig. 1. Then X is shape equivalent to S^1 but X is not CE equivalent to S^1.

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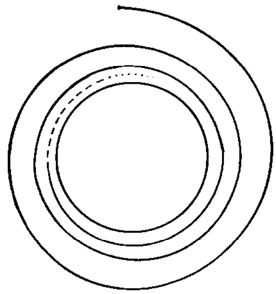

Fig. 1. Ferry's example.

Notice that X is 1-dimensional and planar but is neither path connected nor locally connected. This raises the question of whether or not such an example could exist if X were path connected or locally connected. The following two theorems show that the answer is negative in the correct settings.

Theorem 1.0. *If X is a path connected planar continuum, then X is CE equivalent either to a finite bouquet of circles or to the Hawaiian earring.*

Corollary 1.1. *Suppose X and Y are path connected planar continua. Then X and Y are CE equivalent if and only if X and Y are shape equivalent.*

Theorem 2.0. *For every locally connected, 1-dimensional continuum X there exists a CE map $f: X \to Y$ where Y is planar.*

Corollary 2.1. *Suppose X and Y are locally connected, 1-dimensional compacta. Then X and Y are CE equivalent if and only if X and Y are shape equivalent.*

While locally connected, 1-dimensional continua are CE equivalent if and only if they are shape equivalent, the same is not true of locally connected continua in general. The following is a generalization of Ferry's example.

Example 3.0. For every $n \geq 1$ there exists an n-dimensional, LC^{n-2} continuum $X \subset \mathbb{R}^{n+1}$ such that $\mathrm{Sh}(X) = \mathrm{Sh}(S^1)$ but X is not CE equivalent to S^1.

There are some questions left unanswered by the preceding results. Since the example is only locally connected through dimension $n-2$, it may be possible to generalize Corollary 2.1 to cover the locally $(n-1)$-connected case. Specifically, if X and Y are n-dimensional, LC^{n-1} continua such that $\mathrm{Sh}(X) = \mathrm{Sh}(Y)$, then are X and Y CE equivalent? Another question is whether or not an example like that in Example 1 could exist for 1-UV continua.

We collect below some of the definitions and notation which will be used in the remainder of this paper. A *compactum* is compact metric space and a *continuum* is a connected compactum. A map $f: X \to Y$ is *cell-like* (or CE) if f is a surjection and $f^{-1}(y)$ has the shape of a point for each $y \in Y$. We assume that the reader is familiar with the fundamentals of shape theory (as contained in [3], for example). A space X is LC^k, $k \geq 0$, if for every $x \in X$ and for every neighborhood U of x there exists a neighborhood V of x in U such that each map of S^k into V is null-homotopic in U. We remark that a compactum is LC^0 if it is locally connected. A *bouquet of circles* is the wedge of a finite number of disjoint copies of S^1. The *Hawaiian earring* is the locally connected planar continuum which is the union of the circles $C_n = \{(x, y) | x^2 + (y - 1/n)^2 = 1/n^2\}$, $n = 1, 2, 3, \ldots$.

1. Planar continua

Before beginning the proof of Theorem 1.0 we review a standard definition. Suppose X is a continuum in the plane $\mathbb{R}^2$ and U is a component of $\mathbb{R}^2 - X$. A point x in the frontier of U, $\mathrm{Fr}(U)$, is *accessible* from U if there exists an arc $A \subset U \cup \{x\}$ which has x as an endpoint. It is easy to see that the set of accessible points is dense in the frontier of U.

Proof of Theorem 1.0. Suppose $X \subset \mathbb{R}^2$ is path connected. Let U_0 be the unbounded component of $\mathbb{R}^2 - X$ and let $U_1, U_2, \ldots$ be the bounded components. We will assume that there are infinitely many components U_i and prove that X is CE equivalent to the Hawaiian earring. If there were only finitely many components, a similar proof would show that X is CE equivalent to a bouquet of circles.

For each $i \geq 0$, pick a point $a_i \in \mathrm{Fr}(U_i)$ which is accessible from U_i. Let A_1 be an arc in X from a_0 to a_1. Let B_2 be an arc in X from a_0 to a_2 and let b_2 be the last point of B_2 which lies in A_1. We define A_2 to be the arc which is constructed by following A_1 from a_0 to b_2 and then following B_2 from b_2 to a_2. Now let B_3 be an arc in X from a_0 to a_3 and let b_3 be the last point of B_3 which lies in $A_1 \cup A_2$. Then A_3 is defined to be the arc which is constructed by following either A_1 or A_2 (whichever contains b_3) from a_0 to b_3 and then following B_3 to a_3. This construction is continued inductively and results in a sequence $A_1, A_2, A_3, \ldots$ of arcs with the following two properties: A_i joins a_0 to a_i and no A_i crosses over an A_j.

We next describe how to thicken X up along each of the arcs A_i to form a new planar continuum Z. This continuum Z will have the property that there are cell-like maps $f: Z \to X$ and $g: Z \to H$, where H is the Hawaiian earring.

Since a_0 is accessible from U_0 and a_1 is accessible from U_1, we can extend A_1 to an arc $\tilde{A}_1$ which contains A_1 in its interior such that $\tilde{A}_1 \cap X = A_1$ and $\tilde{A}_1 \subset A_1 \cup U_0 \cup U_1$. By the Schoenflies theorem, $\tilde{A}_1$ is locally flat and so there exists an embedding $h_1: \tilde{A}_1 \times [0, 1] \to \mathbb{R}^2$ such that $h_1(x, 0) = x$ for every $x \in \tilde{A}_1$. Let $p_1: \mathbb{R}^2 \to \mathbb{R}^2$ be a cell-like map which has the arcs $h_1(\{x\} \times [0, 1])$ as its only nontrivial point inverses and has its support in a small neighborhood of $h_1(\tilde{A}_1 \times [0, 1])$. Notice that $X_1 = p_1^{-1}(X)$ is a path connected continuum and that there is a natural one-to-one, onto map F_1: $(X - A_1) \cup A_1 \times [0, 1] \to X_1$. The map is natural in that $F_1(x) = p_1^{-1}(x)$

for $x \in X - A_1$ and $p_1 F_1(a, t) = a$ for $(a, t) \in A_1 \times [0, 1]$. Also notice that the compact subset $X_1 - h_1(\tilde{A}_1 \times (0, 1))$ is connected and has one complementary domain fewer than X does in the sense that

$$p_1^{-1}(U_0) \cup h_1(\tilde{A}_1 \times (0, 1)) \cup p_1^{-1}(U_1)$$

is a connected open subset of the complement of $X_1 - h_1(\tilde{A}_1 \times (0, 1))$. See Fig. 2.

Since A_2 does not cross A_1, there is an arc $\tilde{A}_2 \subset X_1 - F_1(A_1 \times (0, 1))$ such that $p_1(\tilde{A}_2) = A_2$. Extend $\tilde{A}_2$ a little into the open sets $p_1^{-1}(U_0)$ and $p_1^{-1}(U_2)$ and find an embedding $h_2 : \tilde{A}_2 \times [2, 3] \to \mathbb{R}^2$ such that $h_2(x, 2) = x$ for every $x \in \tilde{A}_2$. Let $p_2 : \mathbb{R}^2 \to \mathbb{R}^2$ be a cell-like map which shrinks out the arcs $h_2(\{x\} \times [2, 3])$ and let $X_2 = p_2^{-1}(X_1)$. As before, there is a naturally defined map F_2: $(X - (A_1 \cup A_2)) \cup (A_1 \times [0, 1] \cup A_2 \times [2, 3]) \to X_2$. This map is one-to-one on each of the components of its domain and $F_2(A_1 \times (0, 1)) \cap F_2(A_2 \times (2, 3)) = \emptyset$. By making the diameter of the fibers $h_2(\{x\} \times [2, 3])$ small we can make $F_2|(X - (A_1 \cup A_2)) \cup (A_1 \times [0, 1])$ as close to $F_1|(X - (A_1 \cup A_2)) \cup (A_1 \times [0, 1])$ as we like.

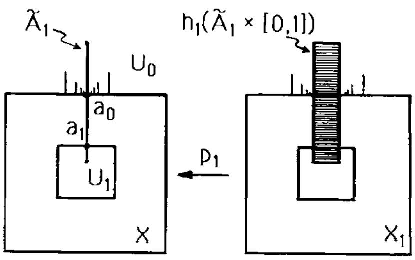

Fig. 2.

This construction is now continued inductively. We construct the cell-like map p_3 by thickening up A_3 and define a natural map

$$F_3 : (X - (A_1 \cup A_2 \cup A_3)) \cup (A_1 \times [0, 1] \cup A_2 \times [2, 3] \cup A_3 \times [4, 5]) \to X_3$$

as before. The construction is done in such a way that the sequence $\{F_n\}$ converges to an embedding on $X - \bigcup A_i$ and on $A_j \times [2j-2, 2j-1]$ for each j and so that $\lim_{n\to\infty} F_n(y_1) = \lim_{n\to\infty} F_n(y_2)$ if and only if there exists an n such that $F_n(y_1) = F_n(y_2)$. Let $F : (X - \bigcup A_i) \cup (\bigcup (A_i \times [2i-2, 2i-1])) \to \mathbb{R}^2$ be defined by $F(x) = \lim_{n\to\infty} F_n(x)$. Define $Z = F((X - \bigcup A_i) \cup (\bigcup (A_i \times [2i-2, 2i-1])))$.

To complete the proof we must define cell-like maps $f : Z \to X$ and g: $Z \to H$. Define f as follows: for $z \in F(X - \bigcup A_i)$, define $f(z) = F^{-1}(z)$ and for $z = F(a, t)$, $(a, t) \in A_i \times [2i-2, 2i-1]$, define $f(z) = a$. This map will be well defined and continuous if $\{F_n\}$ is constructed carefully. Each $f^{-1}(x)$ is a point for $x \in X - \bigcup A_i$ and $f^{-1}(x)$ is an arc if $x \in A_i$ for some i. Specifically, $f^{-1}(x) = \bigcup \{F_i(\{x\} \times [2i-2, 2i-1]) \mid x \in A_i\}$, which is an arc. Thus f is a cell-like map.

We define g by specifying the point preimages. There are two kinds. First, each arc of the form $F(A_i \times \{t\})$, $t \in (2i-2, 2i-1)$, is mapped to a point by g. There is only one preimage of the second kind:

$$Z_0 = Z - \bigcup F(A_i \times (2i-2, 2i-1)).$$

This last set is cell-like because it is a connected subset of the plane which does not separate the plane. (The construction of Z was specifically designed to make enough channels in Z so that Z_0 does not separate $\mathbb{R}^2$.) We must check that $g(Z) = H$. This is so because $g(Z_0)$ is a point and each of the sets $F(A_i \times [2i-2, 2i-1])$ is mapped onto a loop at $g(Z_0)$. The diameter of these loops decreases because we thicken up the arcs A_i by smaller and smaller amounts. □

Proof of Corollary 1.1. Let X and Y be path connected, planar continua. It is a standard result that if X and Y are CE equivalent, then X and Y are shape equivalent [8]. (By a theorem of Kozlowski, this is true even if the connecting compactum in the CE equivalence is infinite dimensional [3, Theorem 10.4.5].) Alternatively, a proof preferred by the referee is to use the Vietoris–Begle Theorem to conclude that $\bar{H}^1(X) \approx \bar{H}^1(Y)$, to apply duality to see that $\mathbb{R}^2 - X$ and $\mathbb{R}^2 - Y$ have the same number of components, and to invoke Borsuk's classification of planar shapes [1] to certify that X and Y are shape equivalent.

Conversely, suppose X and Y are shape equivalent. By Theorem 1.0, X is CE equivalent to X_1 and Y is CE equivalent to Y_1 where each of X_1 and Y_1 is either a bouquet of circles or the Hawaiian earring. But $\mathrm{Sh}(X) = \mathrm{Sh}(Y)$ implies that $\mathrm{Sh}(X_1) = \mathrm{Sh}(Y_1)$ which, in turn, implies that $X_1 = Y_1$ and so X and Y are CE equivalent. □

2. Locally connected 1-dimensional continua

This section contains the proofs of Theorem 2.0 and Corollary 2.1. We begin with some definitions and then prove a lemma which gives us a convenient inverse limit representation for a locally connected, 1-dimensional continuum. The definitions are standard and can also be found in [7].

Let X be a compactum. An *open cover* α of X is a finite collection of nonempty open sets whose union is all of X. The cover α has *order* $\leq n$ if no $(n+2)$ elements of α have a point in common. Associated with any cover α there is an abstract polyhedron $N(\alpha)$ called the *nerve* of α. The vertices of $N(\alpha)$ are the elements of α and the simplex spanned by a collection of vertices is in $N(\alpha)$ if and only if the open sets corresponding to those vertices have nonempty intersection. There is a natural map $\pi : X \to N(\alpha)$, called the *barycentric map*, such that $\pi(x)$ belongs to the interior of the unique simplex spanned by the collection of all elements from α containing x. See [7, p. 70] for a definition of π.

Lemma 2.2. *Suppose X is a 1-dimensional, locally connected continuum. Then for every open cover α of X there exists a refinement β of α such that β has order ≤ 1, each element of β is connected, $\pi : X \to N(\beta)$ is onto, and $\pi^{-1}(v)$ is connected for each vertex $v \in N(\beta)$.*

Proof. Begin with a refinement γ of order ≤ 1. Say $\gamma = \{V_1, V_2, \ldots, V_n\}$. For each i, let $\text{core}(V_i) = V_i - \bigcup\{V_j \mid j \neq i\} = X - \bigcup\{V_j \mid j \neq i\}$. By discarding redundant V_i's, if necessary, we may assume that each core is nonempty. Then $\text{core}(V_i)$ is compact and is precisely $\pi^{-1}(V_i)$. The strategy of the proof is to modify the open sets in γ to make each core connected. This is done in three stages: first we make the total number of components of the cores finite, then we make each element of the cover connected, and finally we work on the cores one at a time to make each core have only one component.

Let $\varepsilon = \min\{\text{dist}(\text{core}(V_i), X - V_i)\}$. For each $x \in X$ there exists a compact, connected set C_x such that $x \in \text{int}\, C_x$ and $\text{diam}(C_x) < \frac{1}{3}\varepsilon$. There are finitely many, $C_1, C_2, \ldots, C_m$, of these C_x whose union covers X. We define $X_i = \cup \{C_j \mid C_j \cap \text{core}(V_i) \neq \emptyset\}$ and $U_i = V_i - \bigcup\{X_j \mid j \neq i\}$. The collection $\{X_i\}$ consists of pairwise disjoint compact sets with $X_i \subset V_i$; moreover, $\{U_i\}$ covers X, since for $x \in V_i$ either x lies in some X_j in which case $x \in U_j$ or x belongs to no X_j in which case $x \in U_i$. The cover $\{U_i\}$ has the property that $\text{core}(U_i) = X_i$. Clearly $X_i \subset \text{core}(U_i)$. If $x \in U_i - X_i$, then x belongs to no $\text{core}(V_j)$. Thus $x \in V_i \cap V_k$ for some k implying x is not in $\text{core}(U_i)$. Observe that the total number of components of these cores is at most m.

Now let β be the cover whose elements are components of elements of $\{U_i\}$. Then β is a refinement of α and has order ≤ 1; we continue to denote the elements by U_i. By again eliminating any redundant sets, we make β finite. We claim that each core is then nonempty and that the total number of components of the cores is still at most m. It is clear that each core is nonempty because if $\text{core}(U_i) = \emptyset$ for some i, then we eliminate that U_i. In order to prove the claim that the total number of components of cores is still at most m, we consider a connected open set U_j which is redundant. Then U_j is covered by the open sets $U_j \cap U_i$, $j \neq i$. These sets are pairwise disjoint because the order of β is ≤ 1 and a finite number cover U_j by compactness. Thus, the fact that U_j is connected means that at most one $U_j \cap U_i$ is nonempty; i.e. there exists an i such that U_j is entirely contained in U_i and U_j intersects no other element of β. It must be the case that $\text{Cl}(U_j) \cap \text{core}(U_i) \neq \emptyset$ since U_i is connected. We can now see exactly what will happen to the cores when U_j is eliminated: they will all remain unchanged except for $\text{core}(U_i)$ which will be increased by having the connected set $\text{Cl}(U_j)$ added to it. The number of components of $\text{core}(U_i)$ will not increase when this happens because of the fact that $\text{Cl}(U_j) \cap \text{core}(U_i) \neq \emptyset$. This completes the proof of the claim.

Consider $\text{core}(U_1)$. Choose an arc A in U_1 such that $\text{core}(U_1) \cup A$ is connected. Now let $W_1 = U_1$, and $W_i = U_i - A$ for $i \geq 2$. Then $\beta' = \{W_i\}$ is again an order 1 cover of X. As before, take components and eliminate redundancy. (The only type of redundancy which will occur at this point is that some components of $U_i - A$, $i \geq 2$, may lie entirely inside U_1 and should therefore be discarded. This will increase the size of $\text{core}(W_1)$ but will not disconnect it.) Now β' has the property that $\pi^{-1}(W_1) = \text{core}(W_1)$ is connected. We next move to W_2 and make its core connected, and then move to W_3, etc. We need to check that this procedure will terminate in a finite number of steps even though there may be more open sets in β' than there were in β. It will, because, if core(U_i) is not connected, then the total number of components of the cores is decreased by the operation of going from β to β'. Specifically, the total number of components of the cores minus the number of cores will strictly decrease and will therefore reach 0 in a finite number of steps.

We have now succeeded in making $\pi^{-1}(V)$ a nonempty connected set for each vertex $V \in N(\beta)$ and in making each element of β connected. If σ is a 1-simplex in $N(\beta)$ spanned by the vertices U and V, then $U \cup V$ is connected and so is $\pi(U \cup V)$. It follows that $\sigma \subset \pi(U \cup V)$ and so π is onto. □

Proof of Theorem 2.0. We will construct a sequence of covers $\alpha_1, \alpha_2, \ldots$ of X, PL maps $f_i: N(\alpha_{i+1}) \to N(\alpha_i)$, and PL CE maps $\lambda_i: N(\alpha_i) \to \mathbb{R}^2$. This construction is done in such a way that $\lambda_i f_i$ is ε_i-homotopic to λ_{i+1}, where $\{\varepsilon_i\}$ is a sequence of positive numbers for which $\sum \varepsilon_i < \infty$. We then apply [9, Proposition 3.2] to conclude that $\{\lambda_i\}$ converges to a CE map f: $X \to f(X) \subset \mathbb{R}^2$. (See also Fig. 3.)

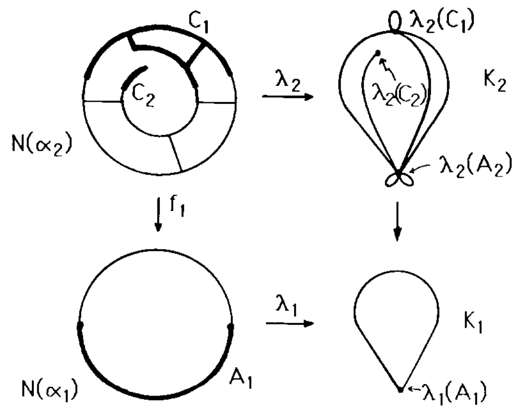

Fig. 3.

Let α_1 be a cover of X which satisfies the conclusions of Lemma 2.2 and let $\pi_1: X \to N(\alpha_1)$ denote the barycentric map. Now $N(\alpha_1)$ is a connected 1-dimensional polyhedron with a fixed triangulation; let A_1 denote a maximal tree in that triangulation. We define $\lambda_1: N(\alpha_1) \to \mathbb{R}^2$ by letting $\lambda_1(A_1)$ be a point and mapping the 1-simplices in $N(\alpha_1) - A_1$ onto loops in $\mathbb{R}^2$ which meet only at the point $\lambda_1(A_1)$. Use K_1 to denote $\lambda_1(N(\alpha_1)) \subset \mathbb{R}^2$ and let M_1 be a close regular neighborhood of K_1.

Now let α_2 be a refinement of α_1 which satisfies the conclusions of Lemma 2.2 and has much smaller mesh. (Just how small the mesh should be will become apparent during the construction below.) Of course π_2 denotes the barycentric map $X \to N(\alpha_2)$.

We define $f_1: N(\alpha_2) \to N(\alpha_1)$ as follows. For each $U \in \alpha_2$ pick a point $u \in U$ and define f_1 of the vertex $U \in N(\alpha_2)$ by $f_1(U) = \pi_1(u)$. This defines f_1 on all the vertices of $N(\alpha_2)$. Now suppose σ is the simplex in α_2 joining the vertices U and V. Extend f_1 to σ by mapping σ onto the shortest path in $N(\alpha_1)$ from $f_1(U)$ to $f_1(V)$. If the mesh of α_2 is small enough, this shortest path will be well-defined. Notice that we can make $\pi_2 f_1$ be as close as we wish to π_1 by decreasing the mesh of α_2. This completes the definition of f_1.

We next define a PL CE map $\lambda_2: N(\alpha_2) \to M_1$. Notice that the fact that $\pi_1^{-1}(V)$ is connected for each vertex V of $N(\alpha_1)$ implies that $\pi_1^{-1}(C)$ is connected for every connected subcomplex (of the natural triangulation) of $N(\alpha_1)$. [Hint: for a 1-simplex $\langle U, V\rangle$ in $N(\alpha_1)$, $\pi_1^{-1}(\langle U, V\rangle) = \text{core}(U) \cup (U \cap V) \cup \text{core}(V)$, and each component of $U \cap V$ meets either core(U) or core(V).] In particular, $\pi_1^{-1}(A_1)$ is connected. Thicken $\pi_1^{-1}(A_1)$ up to a connected open set $W \subset X$. By choosing the mesh of α_2 sufficiently small, we can make sure that $f_1\pi_2(X - W) \cap A_1 = \emptyset$ and thus that $f_1^{-1}(A_1)$ is contained in the connected set $\pi_2(\text{Cl}(W))$. Let B_2 be a connected polyhedral neighborhood of $\pi_2(\text{Cl}(W))$ in $N(\alpha_2)$. By taking W to be a close neighborhood of $\pi_1^{-1}(A_1)$ and taking the mesh of α_2 small, we can make $f_1(B_2)$ be contained in an arbitrarily small neighborhood of A_1. Let A_2 be a maximal tree in B_2. Define $\lambda_2(A_2) = \lambda_1(A_1)$. Make λ_2 map the 1-simplices of $B_2 - A_2$ onto small disjoint open loops in M_1 attached to $\lambda_2(A_2)$. Consider a component C of $N(\alpha_2) - B_2$. We see that C is a finite 1-complex with some nonseparating vertices removed. Let T be a tree in C which contains all the vertices of C. Since there is a 1-simplex σ of $N(\alpha_1)$ such that $f_1(C) \subset \sigma$, we can define $\lambda_2(T)$ to be the midpoint of the loop $\lambda_1(\sigma)$. Let Δ be an open 1-simplex in $C - T$ and let a and b denote the endpoints of Δ. If both a and b are in T, then $\lambda_2(\Delta)$ is defined to be a small loop in M_1 based at $\lambda_2(T)$. Otherwise, $\lambda_2(\Delta)$ is defined to be an arc in M_1 from $\lambda_2(a)$ to $\lambda_2(b)$. (There are two choices for this arc; choose the one which makes $\lambda_2|\text{Cl}\,\Delta$ homotopic to $\lambda_1 f_1$ rel endpoints.) It is not difficult to make these arcs and loops disjoint except for their endpoints. This completes the definition of λ_2. We let K_2 denote $\lambda_2(N(\alpha_2))$ and take M_2 to be a small regular neighborhood of K_2.

This procedure is now continued inductively. The next thing to do is to define α_3 to be a cover whose mesh is much smaller than that of α_2 and then to define $f_2: N(\alpha_3) \to N(\alpha_2)$ in much the same way that f_1 was defined. In order to define $\lambda_3: N(\alpha_3) \to M_2$, we lift the nondegenerate point preimages of λ_2 to disjoint connected sets in $N(\alpha_3)$ and define λ_3 to take each of those connected sets to a point with a number of very small loops attached. Then λ_3 is extended to components of the rest of $N(\alpha_3)$. The result of this construction is an infinite sequence of covers and maps.

A proof like the proof of Brown's Theorem [2] can be used to show that X is homeomorphic to the inverse limit $\lim_{\leftarrow} \{N(\alpha_i), f_i\}$ provided the mesh of α_i gets small fast enough. Furthermore, λ_{i+1} has been constructed in such a way that $\lambda_i f_i$ is homotopic to λ_{i+1} in M_i. The diameter of the track of a point under the homotopy is no larger than the diameter of $\lambda_i(\sigma)$, where σ is a simplex in $N(\alpha_i)$. Those diameters will decrease quickly enough so that [9, Proposition 3.2] applies to give the CE map $f: X \to f(X) \subset \mathbb{R}^2$ which we seek. □

Proof of Corollary 2.1. Let X and Y be locally connected, 1-dimensional compacta. If X and Y are CE equivalent then they are shape equivalent just as in the proof of Corollary 1.1.

Conversely, suppose X and Y are shape equivalent. We first consider the case in which X and Y are connected. In that case we can apply Theorems. 1.0 and 2.0 to conclude that X is CE equivalent to X_1 and Y is CE equivalent to Y_1 where each of X_1 and Y_1 is either a bouquet of circles or the Hawaiian earring. But $\mathrm{Sh}(X)=\mathrm{Sh}(Y)$ implies $\mathrm{Sh}(X_1)=\mathrm{Sh}(Y_1)$ which, in turn, implies that $X_1=Y_1$. Any locally connected compactum has at most a finite number of components, so if X and Y are not connected we may apply the proof above to the components. □

Remark. Suppose X and Y are locally connected, 1-dimensional continua which are shape equivalent. The proofs of Theorems 1.0 and 2.0 can actually be made to yield a sequence of spaces and cell-like maps connecting X and Y in which each space is locally connected and 1-dimensional. Specifically, there is a sequence

$$X \to f(X) \leftarrow Z_1 \to X_1 = Y_1 \leftarrow Z_2 \to g(Y) \leftarrow Y$$

in which each map is a CE map and each space is a locally connected, 1-dimensional continuum. The spaces $f(X)$ and $g(Y)$ are those constructed in the proof of Theorem 2.0 and obviously have the properties specified, while X_1 and Y_1 are both either bouquets of circles or Hawaiian earrings. Thus we need only check that Z_1 and Z_2 can be locally connected and 1-dimensional. In general, the continua constructed in the proof of Theorem 1.0 will be 2-dimensional, but $f(X)$ and $g(X)$ are nice enough so that this can be avoided. Both contain many points which locally separate. (Each vertex of K_i is such a point.) Thus, instead of thickening along a sequence of arcs $\{A_i\}$ as we did in the proof of Theorem 1.0, we can get by with thickening $f(X)$ and $g(X)$ up along points. In the case of $f(X)$, the sequence of arcs $\{A_i\}$ is replaced with a sequence of points $\{a_i\}$ having the property that $\mathbb{R}^2-f(X)\cup\{a_i\}$ is connected. Thickening up each of these points to an arc in an appropriate way will not increase the dimension or destroy local connectivity. See Fig. 4 for a picture of

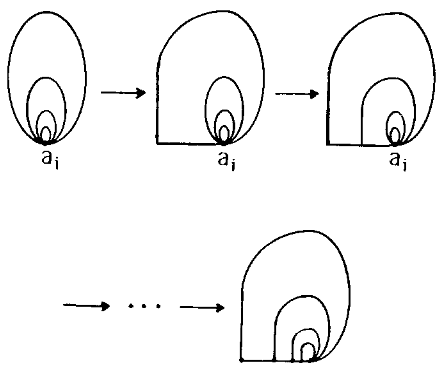

Fig. 4.

how this would be done in case we were dealing with the Hawaiian earring. Notice that the same point a_i can be used more than once to make different complementary domains accessible.

3. An Example

In this section we construct Example 3.0 and prove that it has the properties specified in the Introduction.

Fix $n \geqslant 1$. Let W be the wedge of a null sequence of $(n-1)$-spheres (in case $n=2$, W is just the Hawaiian earring) and let $F: W \to W$ be the embedding which shifts all the $(n-1)$-spheres down one place in the sequence. Our example, X, is the mapping torus of F. Although he did not describe it this way, Ferry's example [5] arises in precisely the same manner when $n=1$. (See Fig. 5.)

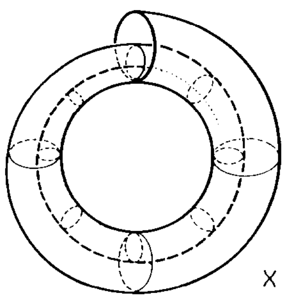

Fig. 5.

Clearly the obvious embedding $\lambda: S^1 \to X$ is a shape equivalence. It should also be clear that X is n-dimensional and LC^{n-2}. The main point is that X and S^1 are not CE equivalent.

Suppose to the contrary that there exist CE maps $f: Z \to X$ and $g: Z \to S^1$ defined on some compactum Z. Consider the universal covers $p: X^* \to X$ and $q: \mathbb{R}^1 \to S^1$ (when $n=2$, X does not have a universal cover, but it does have an infinite cyclic covering space corresponding to the subgroup determined by $\lambda * \pi_1(S^1)$). As in [5], there exist pullbacks

$$\begin{array}{ccc} Z_X & \xrightarrow{f'} & X^* \\ {\scriptstyle p'}\downarrow & & \downarrow{\scriptstyle p} \\ Z & \xrightarrow[f]{} & X \end{array} \qquad \begin{array}{ccc} Z_S & \xrightarrow{g'} & \mathbb{R}^1 \\ {\scriptstyle q'}\downarrow & & \downarrow{\scriptstyle q} \\ Z & \xrightarrow[g]{} & S^1 \end{array}$$

with f' and g' CE surjections and with p' and q' covering maps. The crucial observation is that Z_X and Z_S are the same space, called Z'. Once that has been shown, the contradiction will be transparent, for by the Vietoris–Begle mapping

Theorem, $\bar{H}^{n-1}(X^*) = \bar{H}^{n-1}(\mathbb{R}^1) = 0$, which is impossible because X^* has the homotopy type of the wedge of $(n-1)$-spheres.

There is a geometric way to see why Z_X and Z_S are homeomorphic. Form a new space M by attaching the mapping cylinders of $f: Z \to X$ and $g: Z \to S^1$ together along the common copy of Z. Embed M in the Hilbert cube Q and use the fact that the inclusion-induced $\bar{H}^1(M) \to \bar{H}^1(S^1)$ is an isomorphism to find a neighborhood U of M for which the inclusion $i: S^1 \to M \to U$ induces an injection of fundamental groups. Let $e: U^* \to U$ be the covering space determined by $i_* \pi_1(S^1)$. Now lift the map $iq: \mathbb{R}^1 \to U$ to an embedding $\mathbb{R}^1 \to U^*$ and let M^* denote the component of $e^{-1}(M)$ containing this line. It follows routinely that Z_S is equivalent to $M^* \cap e^{-1}(Z)$ and that f' functions just like the lift of the mapping cylinder collapse. Check that the inclusion $j: X \to M \to U$ also induces an injection of the fundamental groups (for $n > 2$; when $n = 1$ it only injects the part corresponding to the obvious circle in X). Hence, there exists a lift $X^* \to U^*$ of $jp: X^* \to U$ embedding X^* in M^*. The cell-likeness of the maps f and g ensures that $j\lambda: S^1 \to U$ is homotopic to $i: S^1 \to U$. As before, one can see that Z_X is equivalent to $M^* \cap e^{-1}(Z)$.

In conclusion, we remark about the connections with a related topic of UV^k equivalence. Compacta X and Y are said to be *UV^k equivalent* if there exist UV^k maps $f: Z \to X$ and $g: Z \to Y$ defined on some compactum Z. (One calls $f: Z \to X$ a UV^k map if, when Z is regarded as a subset of the Hilbert cube I^∞, for each $x \in X$ and each neighborhood U of $f^{-1}(x)$ in I^∞ there exists another neighborhood V of $f^{-1}(x)$ such that every map $\partial B^{i+1} \to V$ can be extended to a map $B^{i+1} \to U$, $i \in \{0, \ldots, k\}$.) It follows from the Vietoris-Begle Theorem that proper UV^k maps induce isomorphisms of Čech homology groups in dimension i ($i = 0, \ldots, k$). Hence, the argument just given indicates that the example X is not UV^{n-1} equivalent to S^1. (On the other hand, it is not difficult to see that the two spaces are UV^{n-2} equivalent.) This example thereby demonstrates the importance of the UV^2 hypothesis in Ferry's result [6] that shape equivalent UV^2 compacta are UV^k equivalent for all k. As the referee has pointed out, this raises another question: is every LC^k compactum X that is shape equivalent to S^1 also UV^k equivalent to S^1?

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