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STRUCTURAL INTERACTIONS OF THE RECURSIVELY ENUMERABLE T- AND W-DEGREES

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1. Introduction and notation

Recently, quite a number of results concerning r.e. sets and their degrees have been obtained by analysing structural interactions of the r.e. Turing (T-) degrees and r.e. weak truth table (W-) degrees. Such results include degree-theoretic splitting properties of r.e. sets (Ambos-Spies [1, 2], Ambos-Spies and Fejer [4], Lerman and Remmel [24, 25], Downey and Welch [11]), degree-theoretic properties of effective model theory (Downey, Remmel and Welch [10], Downey and Remmel [9]), and structural properties of the upper semilattice of r.e. T-degrees $\mathcal{R}$, (Ladner and Sasso [22], Stob [33], Ambos-Spies [1, 2], Downey and Welch [11], and Ambos-Spies and Soare [6]).

One archetypal example is the use of contiguous r.e. degrees. These are r.e. T-degrees consisting of only one r.e. W-degree. Here, for example, Ladner and Sasso showed in [22]:

(i) Every nonzero r.e. W-degree has the W-anticupping property. (That is, if $A$ is r.e. and nonrecursive, there is an r.e. set $\emptyset <_T B <_W A$, such that for all r.e. $C$ if $C \oplus B =_W A$ then $C =_W A$.)

(ii) Therefore, every contiguous r.e. degree has the T-anticupping property, and

(iii) every nonzero r.e. T-degree has an r.e. nonzero contiguous predecessor.

This structural interaction therefore allowed them to conclude that every nonzero r.e. T-degree bounds a nonzero T-degree with the anticupping property. One feature of the r.e. W-degrees is that they form a distributive upper

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semilattice $\mathcal{W}$. This was first noticed by Lachlan (cf. [33]) and in this paper this property will be called:

**Lachlan's Lemma.** If $A \leq_w B_1 \oplus B_2$ are r.e. sets, then there is an r.e. splitting $A_1 \sqcup A_2 = A$ of $A$ with $A_i \leq_w B_i$ for $i = 1, 2$.

This property, and various other technical features of W-degrees, often make it possible to give various structural properties of $\mathcal{W}$. It is often then possible to exploit interactions of $\mathcal{R}$ and $\mathcal{W}$ like contiguous degrees to obtain information about $\mathcal{R}$.

However, this is only possible when one can find useful interactions of $\mathcal{R}$ and $\mathcal{W}$. Apart from contiguous degrees, some useful interactions include the 'semi-contiguous' degrees of [4], Lerman and Remmel's UWP.

Another important reason for studying these interactions comes from 'locally computable' effective model theory. Let $Q$ be an r.e. substructure of a recursive model $(M, \mathcal{C})$ where $\mathcal{C}$ is effective on (indices of) finite sets: that is, we can decide whether or not $x \in \mathcal{C}(y_1, \ldots, y_n)$, and so $M$ is 'locally computable'. Now consider an r.e. 'basis' for $Q$, namely a set of generators $B = \{b_0, b_1, \ldots\}$ with $Q = \mathcal{C}(B)$, such that for all $x$, if $x \in \mathcal{C}(b_1, \ldots, b_n)$, then $x \neq b_j$ for $j > n$. It will follow that $B \leq_w Q$.

One of the fundamental questions for $Q$, is "What is the relationship between the degrees of bases (or sets of generators) for $Q$ and deg($Q$)?" The actual interaction may be exceedingly complex (cf. [10]), but nevertheless, many constraints may be placed on $d(B)$ by analysing properties of W-degrees.

Applications include groups with solvable generalized word problems, locally computable rings, algebraically closed fields, theories, recursive vector spaces, etc. (cf. [27]). One result we shall prove later (in Section 4) is that there is an r.e. T-degree $a \neq \emptyset$, such that if $A$ is an r.e. set of degree $a$, there is an r.e. nonrecursive set $B \leq_T A$ such that if $C$ is r.e. and $C \leq_T B$ and $C \leq_w A$, then $C = \emptyset$. This then means that for each of the above systems there is a $Q$ (of degree $a$) with the 'antibasis property'. That is, if $R$ is an r.e. basis of $Q$ with $R \leq_T B$ then $R$ is recursive. There are also similar applications to splitting properties of r.e. sets (cf. [11], [1, 2]).

In view of all of these useful applications, results concerning the relationship between $\mathcal{R}$ and $\mathcal{W}$ are extremely scattered. The goal of this paper is to give some new results concerning the structural inter-relations of $\mathcal{R}$ and $\mathcal{W}$.

Before stating our results, we mention the following non-standard notations: upper case italic letters ($A, B, C \cdots$) will *always* denote r.e. sets. If $A$ is an r.e. set, then $a$ will denote its T-degree and $A$ its W-degree. Also $\cap$ will denote the appropriate infimum operator: thus $A \cap B$ is the intersection of (the r.e. sets) $A$ and $B$, $A \cap B$ is the infimum (if any) of the r.e. W-degrees of $A$ and of $B$ in $\mathcal{W}$, and similarly $a \cap b$ for T-degrees. Hopefully, these conventions will facilitate statements and proofs.
In Section 2 we analyse pairs without infinum. We say a pair of r.e. W-degrees
\((A, B)\) have SNI \(\text{(strong non-infinum property)}\), if for all \(F\) if \(F \leq A, B\), then there
exists \(D \leq A, B\) such that \(d \neq f\). Our main results here are that there is an r.e.
degree \(a\), such that for all \(A \in a\), there is a \(B \in a\) such that \((A, B)\) has SNI.
Furthermore, there is an \(a\) such that for all \(A \in a\), there is a \(B\) with \(0 \leq B \leq A\)
and for all \(C\), if \(c = b\) then \((C, A)\) has SNI.

Consequently, there are r.e. degrees \(a\) containing no \(A\) of greatest or least
W-degree (amongst those of degree \(a\)).

In Section 3 we analyse pairs of intervals of r.e. T- and W-degrees. For
example, it is shown how to construct a pair of nontrivial intervals \(I_1, I_2\) which do
not have an infinum. (That is, for all \(a \in I_1, b \in I_2, a \cap b\) doesn't exist.) These
results may be combined with those of Section 2.

In Section 4 we analyse pairs with infinum. Our main results are

\[\exists A, B \quad (a = b \neq 0 \& A \cap B = 0),\]

\[\exists a \neq 0 \forall B \quad (0 \leq b \leq a \rightarrow \exists D \quad (0 \leq d \leq a \& D \cap B = 0)).\]

(In fact, if \(E\) has the property that \(E \leq B\) and \(e \leq d\), then \(E = 0\).)

This gives the antibasis/antisplitting results mentioned earlier. For the latter
('antisplitting') means that \(A \neq 0\) and there exists \(0 \leq B \leq A\), such that if
\(A_1 \sqcup A_2 = A\) is an r.e. splitting of \(A\), then \(0 \leq A_i \leq B\) implies \(A_i = 0\). We
therefore improve a result of Lerman and Remmel [24] to show that there is a
completely antisplitting r.e. degree: a degree \(a \neq 0\) such that every r.e. set of
degree \(a\) has the antisplitting (and antibasis) property.

All sets with the antisplitting property constructed thus far are capable. We
shall also extend our proof technique to refute the conjecture that all such sets
are capable, by constructing a complete r.e. set with the antisplitting property.

It is assumed that the reader is already familiar with W-degrees and
capping/cupping properties of \(\mathcal{R}\). Our notation is mostly consistent with Soare
[30, 31, 32]. For degrees, upper case greek letters \((\Phi, \Psi, \Gamma, \Xi, \Omega, \ldots\) always
denote T-functionals, and such letters with hats \((\hat{\Phi}, \hat{\Psi}, \hat{\Gamma}, \ldots\) denote W-
functionals with use functions the corresponding lower case greek letter. Thus,
the use function of \(\hat{\Phi}\) is \(\phi\). This is standard from [22]. \(\sqcup\) will denote splitting so
that \(A = B \sqcup C\) means \(A = B \cup C\) and \(B \cap C = \emptyset\). Finally, we have the convention
that all computations, etc. at stage \(s\) of a construction are bounded by \(s\). For any
other terminology, we refer the reader to [30, 31].

2. Pairs without inf

Although our first result is well-known, we give it for motivation.

**Theorem 2.1.** There exist r.e. sets \(A, B\) and \(C\) such that \(A \cap B = C\) and \(a \cap b \neq c\).

**Proof.** Let \(A', B'\) be r.e. sets forming a T-minimal pair, that is, \(a' \cap b' = 0\). Let \(C\)
be an r.e. set such that \( \emptyset \triangleleft_T C \triangleleft_T A' \), and \( c \) is non-branching. Then \( c \cap b' = \emptyset \) and \( c \triangleleft_T a' \), hence putting \( A = A' \oplus C \), \( B = B' \oplus C \), it follows that \( a \cap b \neq \emptyset \).

We claim \( C = A \cap B \). Let \( D \leq_w A, B \). Then by Lachlan's lemma, as \( D \leq_w A' \oplus C \), \( D = H \sqcup K \) with \( H \leq_w A' \) and \( K \leq_w C \). Again as \( H \leq_w B' \oplus C \), \( H = R \sqcup P \) with \( R \leq_w B' \) and \( P \leq_w C \). Now \( R \leq_w B' \) and as \( R \leq_w H \leq_w A' \), we have \( R \leq_w B' \), \( A' \). But \( B' \cap A' = \emptyset \) since \( b' \cap a' = \emptyset \). Hence \( r = 0 \). Therefore \( D \leq_w C \). Hence \( A \oplus B = C \). \( \square \)

**Remark.** Downey has shown that it is possible to have \( a \cap b \) not exist in 2.1. This result will appear elsewhere.

Our next result extends 2.1, and moreover, explores the structure of r.e. \( W \)-degrees within an r.e. \( T \)-degree. Although we later improve on this result, it is given as a basis for these later improvements.

**Theorem 2.2** (i) There exist r.e. sets \( A, B \) such that \( A = -T B \) and \( A \cap B \) doesn't exist.

(ii) In fact, we can ensure that if \( C \leq_w A, B \) then there exists \( D \) such that \( D \leq_w C \).

**Proof.** Let \( \langle \Phi_e, \hat{r}_e, W_e \rangle \) list triples consisting of pairs of W-reductions \( \Phi_e, \hat{r}_e \), (with use functions \( \phi_e, \gamma_e \) respectively) together with an r.e. set \( W_e \). We shall build r.e. sets \( A, B, V_e \) for \( e \in \omega \) satisfying the requirements

\[ R_{(e,i)}: \text{ If } \Phi_e(A) = \hat{r}_e(B) = W_e \text{ then } V_e \leq_w B, A \text{ and } V_e \neq \Phi_i(W_e). \]

Here \( \Phi_i \) denotes the \( i \)-th T-reduction. We shall say that \( R_{(e,i)} \) is *satisfied* if there is \( y \) such that \( \Phi_{i,s}(W_{e,s}; y) \downarrow \) and

(i) \( \Phi_{i,s}(W_{e,s}; y) \neq V_{e,s}(y) \),

(ii) \( r((e,i), s) > \max\{\phi_{e,s}(u), \gamma_{e,s}(u)\} \) where \( \phi_{e,s}(u) \downarrow, \gamma_{e,s}(u) \downarrow \) and \( u = u(\Phi_{i,s}(W_{e,s}; y)) \). (Here \( r((e,i), s) \) denotes a restraint defined in the construction associated with \( R_{(e,i)} \).)

Thus \( R_{(e,i)} \) becomes satisfied when the restraint \( r((e,i), s) \) is preserving a disagreement. Let \( R(k, s) = \max\{r(i, s) : i < k\} \). To meet the \( R_{(e,i)} \) we measure the length of agreement (for \( e \)) via

\[ l(e, s) = \max\{x : \forall y \leq x (\Phi_{e,s}(A_s; y) = \hat{r}_{e,s}(B_s; y) = W_{e,s}(y))\}. \]

A stage \( s \) will be called *\( e \)-expansionary* if \( l(e, s) > m(e, s) \), where \( m(e, s) = \max\{l(e, t) : t < s\} \). Notice: if \( \Phi_e(A) = \hat{r}_e(B) = W_e \), then \( l(e, s) \to \infty \) and there are infinitely many \( e \)-expansionary stages. If \( s \) is \( e \)-expansionary, the *last \( e \)-expansionary stage* is

\[ l_s(e, s) = \begin{cases} \max\{t : t < s \text{ and } t \text{ is } e \text{-expansionary}\} & \text{if } t \text{ exists,} \\ 0 & \text{otherwise.} \end{cases} \]
We shall ensure \( A \preceq_T B \) by a tracing procedure at \( e \)-expansionary stages.

To simplify the overall presentation, if at any stage \( s \), \( x \) is associated with \( R_k \) (as a follower, trace, etc.) and \( R(k, s) \geq y \) for some follower or trace associated with \( R_k \), then \( x \) will be cancelled, as will all followers, traces, etc. associated with requirements of lower priority. Let \( Q(s) \) be the least free number at stage \( s \). \( Q(s) \) thus becomes the first number to exceed all traces, uses etc. at stage \( s \).

We shall say that \( R_{\langle e, i \rangle} \) requires attention at stage \( s + 1 \), if \( s \) is \( e \)-expansionary and \( R_{\langle e, i \rangle} \) is not satisfied at stage \( s + 1 \) and \( \langle e, i \rangle \) is the least number with one of the following options holding:

1. \( R_{\langle e, i \rangle} \) has no follower.
2. \( R_{\langle e, i \rangle} \) has a follower \( x \) which is not realized and
   (i) \( \forall y \leq x \ (\Phi_{i,s}(W_{e,s}; y) \downarrow) \),
   (ii) \( l(e, s) > \max \{ u(\Phi_{i,s}(W_{e,s}; y) \mid y \leq x) \} \).
3. \( R_{\langle e, i \rangle} \) has a realized follower \( x \) and an attack on \( R_{\langle e, i \rangle} \) is in progress.

**Construction, stage \( s + 1 \)**

Adopt the appropriate case.

**Case 1** (Follower assignment): (2.3) holds. Appoint \( x = \langle \langle e, i \rangle, Q(s) \rangle \) as a follower of \( R_{\langle e, i \rangle} \). Set \( r(\langle e, i \rangle, s + 1) = x + 1 \). Set \( V_{e,s+1} = V_{e,s} \).

**Case 2** (Realization): (2.4) holds. Say \( x \) is now realized.

There are now two subcases:

**Subcase (i):** \( \Phi_{i,s}(W_{e,s}; x) \neq 0 \). Set \( r(\langle e, i \rangle, s + 1) = Q(s) + 1 \). \( R_{\langle e, i \rangle} \) is now satisfied since \( x \notin V_{e,s} \).

**Subcase (ii) (Trace appointment):** \( \Phi_{i,s}(W_{e,s}; x) = 0 \). Set \( A_{s+1} = A_s \cup \{ \langle \langle e, i \rangle, Q(s) \rangle \} \) and \( B_{s+1} = B_s \cup \{ \langle \langle e, i \rangle, Q(s) \rangle \} \). Set \( n(\langle e, i \rangle, s) = Q(s) - 1 \) and \( m(\langle e, i \rangle, s) = Q(s) + 1 \). With \( m = m(\langle e, i \rangle, s) \), set \( r(\langle e, i \rangle, s + 1) = m + 1, m + 1 \). Declare \( R_{\langle e, i \rangle} \) to be under attack.

**Case 3** (Attack): (2.5) holds. Let \( m = m(\langle e, i \rangle, s) \) and \( n = n(\langle e, i \rangle, s) \).

There are three subcases:

**Subcase (i):** \( \langle \langle e, i \rangle, n \rangle \notin A_s \). Set \( A_{s+1} = A_s \cup \{ \langle \langle e, i \rangle, n \rangle, \langle \langle e, i \rangle, m \rangle \} \) and \( B_{s+1} = B_s \cup \{ \langle \langle e, i \rangle, m \rangle \} \). Set \( n(\langle e, i \rangle, s + 1) = n \) and \( m(\langle e, i \rangle, s + 1) = m + 1 \). Set \( r(\langle e, i \rangle, s + 1) = (Q(s) + 1, Q(s) + 1) \).

**Subcase (ii):** \( \langle \langle e, i \rangle, n \rangle \in A_s \) and \( x \notin \langle e, i \rangle, n \rangle \). Set \( A_{s+1} = A_s \cup \{ \langle \langle e, i \rangle, m \rangle \} \) and \( B_{s+1} = B_s \cup \{ \langle \langle e, i \rangle, n \rangle, \langle \langle e, i \rangle, m \rangle \} \). Set \( n(\langle e, i \rangle, s + 1) = n - 1 \) and \( m(\langle e, i \rangle, s + 1) = m + 1 \). Set \( r(\langle e, i \rangle, s + 1) = (Q(s) + 1, Q(s) + 1) \).

**Subcase (iii):** \( \langle \langle e, i \rangle, n \rangle \in A_s \) and \( x = \langle \langle e, i \rangle, n \rangle \). Set \( A_{s+1} = A_s \cup \{ \langle \langle e, i \rangle, m \rangle \} \), \( B_{s+1} = \{ \langle \langle e, i \rangle, m \rangle, x \} \cup B_s \) and \( V_{e,s+1} = V_{e,s} \cup \{ x \} \). Set \( r(\langle e, i \rangle, s + 1) = (Q(s) + 1, Q(s) + 1) \). Subcase (iii) completes an attack. \( R_{\langle e, i \rangle} \) will now be (temporarily) met. \( \square \) End of Construction

**Lemma 2.6.** \( A \preceq_T B \).
Proof. Numbers enter \( A \) or \( B \) only if they are followers or traces of followers. Compute the unique \( \langle h, k \rangle \) such that \( z = \langle h, k \rangle \).

Now, \( z \) can enter \( A_{s+1} - A_s \) only if

(i) \( z \in B_{s+1} - B_s \), or

(ii) for some stage \( t < s - 1, \langle h, k + 1 \rangle \in B_{t+1} - B_t \).

If \( \langle h, k + 1 \rangle \not\in B_{s+1} - B_s \) for any \( s \) and (i) doesn’t pertain, then \( z \not\in A \). If \( \langle h, k + 1 \rangle \in B_{s+1} - B_s \) for some \( s \), then find the unique \( s \) where this occurred. If \( z \) is to enter \( A \), then it will do so only for the sake of \( R_h \). At stage \( s \) when \( \langle h, k + 1 \rangle \in B_{s+1} - B_s \), we would have reset \( m(h, s + 1) = m(h, s) + 1 \). Furthermore, if for some \( t > s, z \in A_{t+1} - A_t \), then the construction ensures that \( \langle h, m(h, s + 1) \rangle \in B_{t+1} - B_t \). Hence to determine if \( z \in A_{s+1} - A_s \), find the least stage \( s_1 > s \) such that \( B[\langle h, m(h, s + 1) \rangle + 1] = B_s[\langle h, m(h, s + 1) \rangle] \). Then \( z \in A \leftrightarrow z \in A_{s_1} \). Hence \( A \prec_T B \). \( \square \)

Lemma 2.7. \( B \leq_T A \).

Proof. This is similar to Lemma 2.6 \( \square \)

Lemma 2.8. If \( \hat{\Phi}_e(A) = \hat{\Gamma}_e(B) = W_e \) then \( V_e \leq_w A, B \).

Proof. Under the hypotheses, there are infinitely many \( e \)-expansionary stages. Furthermore \( x \in V_{e,t+1} - V_{e,s} \to s \) is an \( e \)-expansionary stage. Now \( V_e \leq_w B \) by simple permitting. Numbers only enter \( V_e \) during subcase (iii) of Case 3. Thus \( x \in V_{e,t+1} - V_{e,s} \) only if \( x \in B_{s+1} - B_s \) also. Hence given \( x \) find a stage \( s \) where \( B_s[x + 1] = B[x + 1] \). Then \( x \in V_e \leftrightarrow x \in V_{e,s} \). Also \( V_e \leq_w A \) as follows. Find the least stage \( s \) where \( s \) is \( e \)-expansionary and \( A_s[x + 1] = A[x + 1] \). If \( x \notin A_s \), then \( x \notin V_e \): followers must enter \( A \) before they enter \( B \). If \( x \in B_s \), then \( x \) may only enter \( V_e \) at stage \( t + 1 \), where \( t \) is the least \( e \)-expansionary stage \( > s \). Then \( x \in V_e \) iff \( x \in V_{e,e+1} \). \( \square \)

It remains to show that all the \( R_{(e,i)} \) require attention at most finitely often, are met and \( \{ r(\langle e, i \rangle, s) = r(\langle e, i \rangle) \} \) exists. Find a stage \( s_0 \) where, by induction, all the \( R_j \) for \( j < \langle e, i \rangle \) cease to act, and \( r(j, s) = r(j, s_0) = r(j) \) for all \( s > s_0 \). At some stage \( s_1 > s_0 \), \( R_{(e,i)} \) will be appointed a follower \( x \). If (2.4) never holds after stage \( s_1 \), then \( R_{(e,i)} \) will be met by fiat and \( r(\langle e, i \rangle, s_i) = r(\langle e, i \rangle) \).

Thus let \( s_2 > s_1 \) be the least stage where (2.4) holds. By restraints, if Case 2(i) applies, then \( R_{(e,i)} \) becomes met at this stage, and \( r(\langle e, i \rangle, s_2 + 1) = r(\langle e, i \rangle) \) since we will preserve the computation,

\[
(2.9) \quad \Phi_{i,s_2}(W_{e,s_2}; x) \neq 0 = V_{e,s_2}(x).
\]

We shall now argue that an attack succeeds. Let \( u = u(\Phi_{i,s_2}(W_{e,s_2}; x)) \). Now as (2.4) holds at stage \( s_2 \) we know

\[
s_2 > l(e, s_2) > u.
\]
Also, \( r(\langle e, i\rangle, s_2 + 1) > s_2 > \max\{\phi_{e,s_2}(u), \gamma_{e,s_2}(u)\} \). A simple induction will establish the following lemma which shows that this attack started at stage \( s_2 \) will succeed.

Lemma 2.10. Suppose \( \Phi_e(A) = \hat{\Gamma}_e(B) = W_e \). Let \( t_1, t_2 \) be \( e \)-expansion stages with \( t_1, t_2 > s_2 \) and \( t_2 = ls(e, t_1) \). Then

\[
\text{card}\{(A_{t_1} \cup B_{t_1})[s_2] - (A_{t_2} \cup B_{t_2})[s_2]\} \leq 1.
\]

Proof. Straightforward induction. □

By Lemma 2.10, this means that at least one of the \( \Phi_{e,t_2}(A_{t_2}; y) \) or the \( \hat{\Gamma}_{e,t_2}(B_{t_2}; y) \) (for all \( y < u \)) computations will not change between stages \( t_2 \) and \( t_1 \).

It follows, as in a minimal-pair construction, that \( W_{e,s_2}[u] = W_e[u] \) and hence \( \Phi_i(W_{e,s_2}; x) = \Phi_i(W_e, x) = 0 \). Now, if \( \Phi_e(A) = \hat{\Gamma}_e(B) = W_e \), then at each \( e \)-expansionary stage \( \geq s_2 \), we perform one of the actions of Case 3. We do so until \( n(\langle e, i \rangle, s) = x \) and \( x \in A_s \) which must occur after a finite number of attacks. Once Case 3(iii) pertains, we have met \( R_{(e,i)} \), and at such a stage \( t \) we see \( r(\langle e, i \rangle, t + 1) = r(\langle e, i \rangle) \). Finally, if \( \hat{\Phi}_e(A) \neq W_e \) or \( \hat{\Gamma}_e(B) \neq W_e \), there are only finitely many \( e \)-expansionary stages. (After all, we are using \( W \)-reductions.) Now \( R_{(e,i)} \)'s restraint can only change, and \( R_{(e,i)} \) require attention only at \( e \)-expansionary stages. Hence, also in this case we have the result. □

Our next result improves our previous one by producing 'an entire r.e. degree' with the properties of Theorem 2.2. From a technical viewpoint, the roles of the 'm' and 'n' of the previous construction will be taken by certain intervals given by expansion stages.

Theorem 2.11. There exists an r.e. degree \( a \) such that if \( B \in a \), there exists \( C \in a \) with the properties:

(i) \( B \cap C \) doesn't exist.

(ii) In fact, if \( D \leq_w B, C \) then there is an r.e. set \( E \) with \( E \leq_w B, C \) and \( e \notin T_d \).

Proof. We build \( A = \bigcup_s A_s, B_e = \bigcup_s B_{e,s} \) and \( V_{j,e} = \bigcup_s V_{j,e,s} \) such that we satisfy the requirements

\[
R_{(e,i,k)}: \quad \text{If } \Omega_e(A) = W_e \text{ and } \Psi_e(W_e) = A, \text{ then } B_e = TA,
\]

and if \( \Phi_j(W_e) = M_j = \hat{\Gamma}_j(B_e), \) then \( V_{j,e \leq_w W_e} B_e \) and \( \Delta_k(M_j) \neq V_{j,e} \).

Let

\[
L(e, s) = \max\{x \mid \forall y \leq x [\Omega_{e,s}(A_s; y) = W_{e,s}(y) \land \Psi_{e,s}(W_{e,s}; y) = A_s(y)]\},
\]

and \( mL(e, s) = \max\{L(e, t) \mid t < s\} \). Let

\[
l(e, j, s) = \max\{x \mid \forall y \leq x [\Phi_{j,s}(W_{e,s}; y) = M_{j,s}(y) \land \hat{\Gamma}_{j,s}(B_{e,s}; y) = M_{j,s}(y)
\]

\[&\land \phi_{j,s}(y), \gamma_{j,s}(y) < L(e, s)]\}.
\]
We shall say $R_{(e,j,k)}$ is satisfied at stage $s$ if for some $y$,

(a) $\Delta_{k,s}(M_{j,s}; y) \downarrow$,
(b) $\Delta_{k,s}(M_{j,s}; y) \notin V_{i,e,s}(y)$,
(c) $r(\langle e, j, k \rangle, s) > u(\Omega_{e,s}(A_s; m))$ where
(d) $m = \max \{ u(\Psi_{j,s}(W_{e,s}; y)) : y \leq u_1 \}$ with
(e) $u_1 = \max \{ \phi_{j,s}(u), \gamma_{j,s}(u) \}$ and
(f) $u = u(\Delta_{k,s}(M_{j,s}; y))$.

**Remark.** That is, $r(\langle e, j, k \rangle, s)$ is currently preserving a disagreement by restraining $A_s$ and $B_{e,s}$ (for each $e$).

A stage $s$ is called $e$-expansionary if $L(e, s) > mL(e, s)$, where $mL(e, s) = \max \{ L(e, t) : t < s \}$. Similarly, we may define $(e, j)$-expansionary $ls(e, s)$, etc.

A requirement $R_{(e,j,k)}$ requires attention at stage $s + 1$, if it is the unsatisfied requirement of highest priority such that one of the following hold:

1. **(2.12)** $R_{(e,j,k)}$ has no follower and
   (i) $s$ is an $(e, j)$-expansionary stage,
   (ii) $l(e, j, s) > R(\langle e, j, s \rangle, s)$.

2. **(2.13)** $R_{(e,j,k)}$ has an e-frozen follower $x$ such that
   (i) $s$ is $e$-expansionary,
   (ii) $L(e, s) > u(\Omega_{e,s}(A_s; z))$ where
   (iii) $z = \max \{ u(\Psi_{e,s}(W_{e,s}; y)) : y \leq R(\langle e, j, k \rangle, s) \}$.

3. **(2.14)** $R_{(e,j,k)}$ has an unrealized follower $x$ such that
   (i) $x$ is not $e$-frozen,
   (ii) $s$ is an $(e, j)$-expansionary stage,
   (iii) $l(e, j, s) > u(\Omega_{e,s}(A_s; m))$ where
   (iv) $m = \max \{ u(\Psi_{e,s}(W_{e,s}; y)) : y \leq M \}$, with
   (v) $M = \max \{ \gamma_{j,s}(u), \phi_{j,s}(u), R(\langle e, j, k \rangle, s) \}$,
   (vi) $u = u(\Delta_{k,s}(M_{j,s}; x))$ and
   (vii) $\Delta_{k,s}(M_{j,s}; x) \downarrow$.

4. **(2.15)** $R_{(e,j,k)}$ has a realized follower $x$ and
   (i) $R_{(e,j,k)}$ is under attack,
   (ii) $s$ is $(e, j)$-expansionary.

**Construction, stage $s + 1$**

**Remark.** Again we use the conventions of the first construction.

**Case 1:** (2.12) holds. Appoint $x = (\langle e, j, k \rangle, Q(s))$ as a follower for $R_{(e,j,k)}$.

Set $r(\langle e, j, k \rangle, s + 1) = (x + 1, x + 1)$. Declare $R_{(e,j,k)}$ as e-frozen.

**Case 2:** (2.13) holds. $R_{(e,j,k)}$ is no longer e-frozen. Assign the primary trace
$\Gamma(x)$ to $x$ by setting

$$\Gamma(x) = \langle \langle e, j, k \rangle, Q(s) \rangle.$$ 

Now set $r(\langle e, j, k \rangle, s + 1) = \langle \Gamma(x) + 1, \Gamma(x) + 1 \rangle$.

Case 3: (2.14) holds. $x$ is now realized. There are two subcases.

Subcase (i): $\Delta_{k,s}(M_{j,s}; x) \neq 0$. In this case set $r(\langle e, j, k \rangle, s + 1) = \langle Q(s) + 1, Q(s) + 1 \rangle$. $R_{\langle e, j, k \rangle}$ now becomes (temporarily) met.

Subcase (ii): $\Delta_{k,s}(M_{j,s}; x) = 0$. Begin an attack on $R_{\langle e, j, k \rangle}$ as follows: Set

$$m = m(\langle e, j, k \rangle, s + 1) = Q(s) + 1,$$

$$n(\langle e, j, k \rangle, s + 1) = Q(s) - 1,$$

$$A_{s+1} = A_s \cup \{\langle\langle e, j, k \rangle, Q(s)\rangle\}.$$ 

Finally, for all $f \in \omega$, set

$$B_{f,s+1} = B_{f,s} \cup \{\langle\langle e, j, k \rangle, Q(s)\rangle\},$$

and set

$$r(\langle e, j, k \rangle, s + 1) = \langle m + 1, m + 1 \rangle.$$ 

Case 4: (2.15) holds. There are three subcases. Let $m = m(\langle e, j, k \rangle, s)$ and $n = n(\langle e, j, k \rangle, s)$. Set $r(\langle e, j, k \rangle, s + 1) = \langle Q(s), Q(s) \rangle$, and

Subcase (i): $\langle\langle e, j, k \rangle, n \rangle \notin A_s$. Set $A_{s+1} = A_s \cup \{\langle\langle e, j, k \rangle, n \rangle\}$ and $B_{e,s+1} = B_{e,s} \cup \{\langle\langle e, j, k \rangle, m \rangle\}$. For all $f \neq e$, set $B_{f,s+1} = B_{f,s} \cup \{\langle\langle e, j, k \rangle, n \rangle, \langle\langle e, j, k \rangle, m \rangle\}$. Set $m(\langle e, j, k \rangle, s + 1) = m + 1$. Set $n(\langle e, j, k \rangle, s + 1) = n$.

Subcase (ii): $\langle\langle e, j, k \rangle, n \rangle \in A_s$ but $x \neq \langle\langle e, j, k \rangle, n \rangle$. Set $A_{s+1} = A_s \cup \{\langle\langle e, j, k \rangle, m \rangle\}$ and for all $f \in \omega$, $B_{f,s+1} = B_{f,s} \cup \{\langle\langle e, j, k \rangle, m \rangle, \langle\langle e, j, k \rangle, n \rangle\}$. Set $m(\langle e, j, k \rangle, s + 1) = m + 1$ and

$$n(\langle e, j, k \rangle, s + 1) = \begin{cases} x & \text{if } n = \Gamma(x), \\ n - 1 & \text{otherwise.} \end{cases}$$

Subcase (iii): Otherwise. Set $A_{s+1} = A_s \cup \{\langle\langle e, j, k \rangle, m \rangle\}$ set $B_{f,s+1} = B_{f,s} \cup \{\langle\langle e, j, k \rangle, m \rangle, \langle\langle e, j, k \rangle, n \rangle\}$ for all $f \in \omega$ and $V_{j,e,s+1} = V_{j,e,s} \cup \{x\}$. \hfill \Box

End of Construction

We remark that although notationally more complex, the idea remains the same. Restraints ensure that at most one side of the computation will change between $(e, j)$-expansionary stages, and so an inductive argument will show that the analogue of (2.10) holds, once an attack has begun we will have fixed $u(\Delta_{k,s}(M_{j,s}; x))$. The key points we need to show are:

\begin{itemize}
  \item[(2.16)] If $\Omega_e(A) = W_e$ and $\Psi_e(W_e) = A$, then $W_e = \mathbb{T}B_e$, and
  \item[(2.17)] If $\Phi_e(A) = W_e$ and $\hat{\gamma}_e(W_e) = A$, and if $\Phi_e(W_e) = M_j = \hat{\gamma}_j(B_e)$, then $V_{j,e} \leq_w W_e, B_e$.
\end{itemize}
To establish (2.16), it will obviously suffice to show that $A = \bar{T} B_e$, since $W_e = \bar{T} A$.

Now, a member $z$ may enter $A$ (or $B_e$) only if it is a follower, or a descendent of a follower via tracing. Now if $z$ is not a follower or a primary trace by stage $z$, it never will be.

Assume $z \notin A_z$. If $z$ is a follower by stage $z$, find the $R_h$ associated with $z$. If $h \neq (e, i, j)$ for some $i, j$, then $z \in B_e$. If $h = (e, i, j)$ find the least stage $t$ such that either $z$ is cancelled or $z$ is $e$-un-frozen. This is effective as $L(e, s) \to \infty$.

In the former case $z \notin A$. In the latter $I(z)$ is appointed to $z$ and $z \in A$ iff $\Gamma(z) \in B_z$.

If $z$ is not a follower by stage $z$, then it may only enter $A$ as a trace. If $z \notin B_e$, find the unique $(h, k)$ such that $z = (h, k)$. Then $z \in A$ only if $(h, k + 1) \in B_e$. If this is so, find the unique stage $t$ where this occurred. At this stage $m$ was reset as a trace for $z = (h, k)$. Then $z \in A \leftrightarrow m \in B_e$. Hence $A = \bar{T} B_e$.

The other direction is similar to (2.6) and the above, and we leave this to the reader.

Finally, to prove (2.17). Numbers enter $V_{j,e}$ only if the follow $R_{(e,j,k)}$ for some $j, k$ by stage $z$. Thus, to decide if $z$ is a member of $V_{j,e}$ or not, see if $z$ is a follower of some $R_{(e,j,k)}$ for some $k$ by stage $z$. If not $z \notin V_{j,e}$. If so, and $z \notin V_{e,i,z}$, find the stage $t$ where $\Gamma(z)$ is computed, or $z$ gets cancelled. Notice $z \notin V_{j,e,t}$ as it is $e$-frozen in the intervening stages.

Let $u(z) = u(W_{e,i}(z))$. Now, restraints and selection of $\Gamma(z)$ ensure that for the sake of $R_{(e,j,k)}$, $W_{e,s}[u(z)] = W_{e,s}[u(z)]$ until $z \in A_{s+1} = A_e$, or gets cancelled. Thus, to decide if $z \in V_{j,e}$, ask now if $W[u(z)] = W_{e,s}[u(z)]$. If so, then $z \notin V_{j,e}$. If not, find the least stage $s$ with $W_{e,s+1}[u(z)] \neq W_{e,s}[u(z)]$ and $s \geq t$. If $z$ hasn’t been cancelled by this stage, then it can only have been that $z \in A_s + 1 = A_e$ for the sake of $R_{(e,j,k)}$. We know that at the next $(e, j)$-expansionary stage $s_1 > s$, either $z \in V_{j,e,s_1} = V_{j,e,s}$ or $z$ will have been cancelled. Since the computation of $u(z)$ is recursive in $z$ it follows that $V_{j,e} \leq_w W_e$. Finally, $V_{j,e} \leq_w B_e$ by permitting again. This clinches (2.17) and the whole result. \[ \square \]

**Corollary 2.18.** There is an r.e. $T$-degree $a$ containing no maximal or minimal r.e. $W$-degree (that is, amongst the r.e. sets of degree $a$).

**Proof.** Clear.

We can use the technique of the above proof to get several interesting variations. For example, a result of Lerman and Remmel [24] is that there is an r.e. degree $a$, such that if $A \in a$ is r.e., there exists an r.e. set $B \leq_T A$, such that if $C = _T B$ then $C \leq_w A$. In their terminology $a$ is called completely non-UWP. We can improve this as follows.

**Theorem 2.19.** (i) There is an r.e. degree $a$ such that if $A \in a$, there exist $B \leq_T A$ such that if $C = _T B$ then $C \cap A$ doesn't exist.
(ii) **Indeed**, if $D \leq_T C, A$ then there is an r.e. set with $F \leq_A C, A$ and $F \not\leq_T D$.

**Proof.** As many details of this proof are similar to those of (2.11), we shall only sketch some parts. We must build $A = \bigcup_s A_s$, $B_e = \bigcup_s B_{e,s}$ and $V_{e,i,k} = \bigcup_s V_{e,i,k,s}$ to satisfy:

\[ R_h : \text{If } h = \langle e, j, k, f \rangle \text{ then}
\]
\[ (a) \text{ If } \Sigma_e(A) = W_e \text{ and } \Psi_e(W_e) = A \text{ then } B_e \leq_T W_e \text{ and if}
\]
\[ (b) \text{ If } \Phi_k(C_j) = \hat{W}_k \text{ and } \Psi_k(W_k) = C_j \text{ then if}
\]
\[ (c) \text{ If } \Delta_f(W_k) \neq W_{e,j,k}, \text{ then if}
\]
\[ (d) \text{ If } \Delta_f(W_k) \neq W_{e,j,k}.
\]

For the 10-tuples $\langle \Omega_e, W_e, \Sigma_e, \Xi_j, C_j, \Phi_k, \hat{W}_k, A_f \rangle$, there are several lengths of agreement associated with $R_h$.

\[ l(e, s) = \max\{x : \forall y \leq x (\Omega_e,s(A_s;y) = W_{e,s}(y) \& \Psi_e,s(W_{e,s};y) = A_s(y))\}.
\]

\[ L(e, s) = \max\{x : \forall y \leq x (\Omega_e,s(A_s;y) = W_{e,s}(y) \& \Psi_e,s(W_{e,s};y) = A_s(y) \& \Xi_j,s(B_{e,s};y) = C_j,s(y))\}.
\]

and finally,

\[ l(e, j, k, s) = \max\{x : x \leq l(e, j, s) \& \Phi_k,s(C_{j,s};y) = W_{k,s}(y) \& \hat{W}_{k,s}(W_{e,s};y) = W_{k,s}(y))\}.
\]

We may similarly define $mL$, $ml(e, j, s)$ and $ml(e, j, s)$ and so $e$, $(e, j)$- and $(e, j, k)$-expansionary.

One nice aspect of this construction is that we need only change $B_e$ for the sake of $R_{(e,j,k,f)}$ for $j, k, f \in \omega$, and not for $R_{(q, \ldots)}$ with $q \neq e$, since we don't ask that $B_e =_T A$, only that $B_e \leq_T A$. Also for similar reasons, our tracing procedure becomes simpler. An attack, roughly speaking will proceed as follows: we shall put a follower $x$ into $A$ to force a change in $W_e$ at some (recursively chosen) $u(x)$ place. At this time $B_e$-restraints are selected to hold the $j$-computations (b). The attack is $e$-frozen until the next large $e$-expansion stage which is when we can appoint a trace $T(x)$ for $x$. Knowledge that $L(e, s) \rightarrow \infty$ makes this possible. $T(x)$ will be chosen so that if it gets added to $A$ it won't induce a $W_e$-change to upset the $k$-computations (c).

Additional complications will arise because we shall add $x$ to $B_e$ rather than $C_j$, and (after all) we need $V_{e,i,k} \leq_w C_j$. This is overcome by essentially an extra freezing called $(e, j)$-freezing, which, in the same way gives us a 'recursively specified' section of $C_j$ to ask for membership of $V_{e,i,k}$. We shall say $R_h$ is satisfied if we are $r(h, s)$-preserving a disagreement in the sense of (2.11). We give the construction:

We say $R_h = R_{(e, i, j, k, f)}$ requires attention at stage $s + 1$ if it's not currently
satisfied and

(2.20) $R_h$ has no follower and
   
   (i) $s$ is an $(e, j, k)$-expansionary stage,
   (ii) $l(e, j, k, s) > R(h, s)$.

(2.21) $R_h$ has a follower $x$ such that $x$ is $e$-frozen, and
   
   (i) $L(e, s) > u(Q_{e,s}(A_s; z))$ where
   (ii) $z = \max\{u(W_{e,s}; y) : y \leq R((e, j, k, s), s)\}$.

(2.22) $R_h$ has a follower $x$ such that $x$ is $(e, j)$-frozen and (i) and (ii) of (2.21) hold, and furthermore
   
   (i) $s$ is $(e, j)$-expansionary,
   (ii) $l(e, j, s) > \max\{u(Q_{j,s}(B_{e,s}; y)) : y \leq q\}$, where
   (iii) $q = \max\{u(Q_j, s; C_{j,s}; y) : y \leq R((e, j, k, f), s)\}$.

(2.23) $R_h$ has a follower $x$ such that $x$ is unrealized and
   
   (i) $x$ is neither $e$- nor $(e, j)$-frozen,
   (ii) $\Delta_{f,s}(W_{k,s}; x) \downarrow$,
   (iii) $s$ is $(e, j, k)$-expansionary,
   (iv) $l(e, j, k, s) > u(Q_{e,s}(A_s; m))$, where
   (v) $m = \max\{u(W_{e,s}; y) : y \leq M\}$, with
   (vi) $M = \max\{\gamma_{k,s}(U), R((e, j, k, f), s)\}$ such that
   (vii) $U = u(\Delta_{f,s}(W_{k,s}; x))$, and furthermore,
   (viii) $l(e, j, k, s) > u(Q_{j,s}(B_{e,s}; y) : y \leq F)$, where
   (ix) $F = \max\{\phi_{k,s}(U), R((e, j, k), s)\}$, with $U$ as in (vii).

(2.24) $R_h$ has a realized follower $x$ and
   
   (i) $R_h$ is under attack,
   (ii) $s$ is $e$-expansionary,
   (iii) $x$ is waiting, (that is, for a trace).

(2.25) $R_h$ has a realized follower $x$, and
   
   (i) $R_h$ is under attack,
   (ii) $x$ is not waiting,
   (iii) $s$ is $(e, j, k)$-expansionary.

Construction, stage $s + 1$

Case 1: (2.20) holds. Appoint $x = (\langle e, j, k, f \rangle, Q(s))$ as a follower of $R_h$. Set $r(h, s + 1) = \langle x + 1, x + 1 \rangle$. Declare $x$ as $e$-frozen.

Case 2: (2.21) holds. $x$ is no longer $e$-frozen. Assign $T_e(x)$ as $e$-trace for $x$ where $T_e(x) = (\langle e, j, k, f \rangle, Q(s))$. Set $r(h, s + 1) = (T_e(x) + 1, T_e(x) + 1)$.

Case 3: (2.22) holds. $x$ is no longer $(e, j)$-frozen. Assign $T_{e,j}(x)$ as $(e, j)$-trace for $x$ where $T_{e,j}(x) = (h, Q(s))$. Set $r(h, s + 1) = (T_{e,j}(x) + 1, T_{e,j}(x) + 1)$.

Case 4: (2.23) holds. $x$ is now realized. There are two subcases.
Subcase (i): $\Delta_{f,s}(W_{k,s}; x) = 1$. In this case set $r(h, s + 1) = (Q(s), Q(s))$. $R_h$ is temporarily met.

Subcase (ii): $\Delta_{f,s}(W_{k,s}; x) = 0$. Set $A_{s+1} = A_s \cup \{x\}$. Declare an attack on $R_h$ to begin. Set $f(h, s + 1) = (Q(s), Q(s))$. Declare $x$ as waiting.

Case 5: (2.24) holds. $x$ is no longer waiting. Set $B(x) = (h, Q(s))$. ($B(x)$ is a trace with a role similar to the $m$ of (2.11).)

Case 6: (2.25) holds. The attack now finishes. Set $A_{s+1} = A_s \cup \{B(x)\}$. Set $B_{e,s+1} = B_{e,s} \cup \{x\}$. Set $V_{e,j,k,s+1} = V_{e,j,k,s} \cup \{x\}$. Set $r(h, s + 1) = (Q(s), Q(s))$. $R_h$ is now satisfied. \[\square\] End of Construction

We sketch the verification:

Again an inductive argument will show that either $R_h$ is satisfied by a failure of a premise, or becomes satisfied by an attack, and requires attention at most finitely often.

Now, if $\Omega_e(A) = W_e$ and $\Psi_e(W_e) = A$, then $B_e \leq_T A$. Numbers which enter $B_e$ are followers (and $x$ is a follower by stage $x$ if it ever becomes one). Now, if $x \notin A$, then $x \notin B_e$. If $x \in A$, then this can only happen due to the requirement that $x$ follows. Find the stage when $x \in A_{s+1} - A_s$. $x$ is now waiting. Find the least $e$-expansory stage $t \geq s + 1$. Now either $B(x)$ becomes defined or $x$ has been cancelled. If $B(x)$ becomes defined, then $x \in B_e \leftrightarrow B(x) \in A$. Hence $B_e \leq_T A$.

Similarly, by the way the $T_{e,j}(x)$ is defined, if $\Xi_j(B_e) = C_j$ and $\Sigma_j(C_j) = B_e$ and $\Omega_e(A) = W_e$ and $\Psi_e(W_e) = A$, then $V_{e,i,k} \leq_w C_j$. As there are infinitely many $(e, j)$-expansion stages, followers eventually get cancelled or $(e, j)$-traces. As we argued in (2.11), we need only go to the least stage where $C_{i,s}[T_{e,j}(x)] = C_j[T_{e,j}(x)]$. Then either $x \in V_{e,i,k,s}$ or $x \notin V_{e,i,k}$. Finally, in an entirely analogous manner, we may argue for the $T_e(x)$ and hence $V_{e,i,k} \leq_w C_j, W_e$ under the above hypotheses. This concludes our sketch of the proof. \[\square\]

We note that there are nontrivial initial segments of the r.e. $T$-degrees containing no degree in which there is an r.e. set with SNI as above. Recall: $A$ has SNI if $\exists B \leq_T A \forall C = T B \forall D (D \leq_w C, A \rightarrow \exists E (E \leq_w C, A \& E \neq D))$. This follows since Fischer [12] has constructed an initial segment of the r.e. W-degrees containing no pair without infimum. Thus the classification of such degrees would seem difficult. We cannot tie such a to a jump class since the previous construction obviously gives a low degree, and we have

Theorem 2.26. There exists a complete r.e. set with SNI.

Proof. It is routine to blend coding with the obvious modification of Theorem 2.2, and we leave this to the reader. (Use the coding of (4.30).) \[\square\]

3. Pairs of intervals

One of the most interesting questions currently open for the r.e. $T$- or W-degrees is what lattices are dense. For the r.e. $T$-degrees Slaman [35] has
shown that the branching degrees are dense using the very difficult $\emptyset''$ method. In particular, he also showed that the diamond lattice is dense in the r.e. T-degrees. Ambos-Spies (cf. [4]) observed that this implies that $N_5$ (the nondistributive five element lattice) is dense because of Fejer's [36]. It is not known if even the diamond lattice is dense in the r.e. W-degrees.

An even stronger result would be: given $b_1 \mid b_2$, there exist $b'_1, b'_2$ with $b_1 < b'_1 \ll_T b_1 \oplus b_2$ and $b_2 < b'_2 \ll_T b_1 \oplus b_2$ with $b_1 \cap b_2$ existing. We cannot, at this stage, prove or disprove this. We can, however, give the following related construction of a pair of r.e. 'intervals' without infinum. We remark that the construction works equally well in both the T- or W-degrees with the appropriate notation changes. We shall only establish the result for the T-degree case.

**Theorem 3.1.** There exist $a_1 < b_1$ and $a_2 < b_2$ such that for all $x_1, x_2$ with $x_i \in [a_i, b_i]$,

(i) $x_1 \ll_T x_2$,
(ii) $x_1 \cap x_2$ doesn't exist.

**Remark.** We shall write "$[a_1, b_1] \cap [a_2, b_2]$ doesn't exist", in this case.

**Proof.** We shall meet the requirements (building $A_i = \bigcup_s A_{i,s}, B_i = \bigcup_s B_{i,s}$ for $i = 1, 2$):

$R_{5e+1}$: $\Phi_e(B_2) \neq A_1$, $R_{5e+2}$: $\Phi_e(B_1) \neq A_2$,
$R_{5e+3}$: $\Phi_e(A_1) \neq B_1$, $R_{5e+4}$: $\Phi_e(A_2) \neq B_2$,

all by a Friedberg-Muchnik procedure, and the 'non-inf' requirements below:

$R_{5e}$: If $e = (i, j, k)$ then if

(i) $\Phi_i(B_i) = W_i$,  
(ii) $\Gamma_i(W_i) = A_1$,  
(iii) $\Psi_i(B_2) = M_i$,  
(iv) $\Delta_i(M_i) = A_2$,  
(v) $\Sigma_j(W_j) = \Omega_j(M_j) = W_j$, then
(vi) $\Phi_k(W_j) \neq V_{i,j}$, and  
(vii) $V_{i,j} \ll_T W_i, M_i$.

Here $\langle \Phi_i, \Gamma_i, W_i, M_i, \Psi_i, \Delta_i \rangle$, $\langle \Sigma_j, W_j, \Omega_j \rangle$ list the appropriate 6- and 3-tuples respectively. Now the $R_{5e+j}$ for $1 \leq j \leq 4$ are met by the usual Friedberg-Muchnik (cf. [31]) method and we shall assume the reader is familiar with this in a finite injury setting.

To meet the $R_{5e}$ we employ the 'freezing' strategy of Section 2. We measure

$$l(i, s) = \max(x : \forall y \leq x (\Phi_{i,s}(B_{1,s}; y) = W_{i,s}(y)$$

and $\Gamma_{i,s}(W_{i,s}; y) = A_{1,s}(y) \& \Psi_{i,s}(B_{2,s}; y) = M_{i,s}(y)$

and $\Delta_{i,s}(M_{i,s}; y) = A_{2,s}(y))$. 

\[ l(e, j, s) = \max \{x : \forall y (y \leq l(i, s) \& \Sigma_{j, s}(W_{j, s}; y) = \Omega_{j, s}(M_{i, s}; y) = W_{j, s}(y)) \}. \]

This, as in Section 2, allows us to define \( ml, ls, \) etc. It is our intention to ensure that \( B_i \geq_T A_i \) by simple permitting. Thus whenever numbers (followers) enter \( A_{i, s+1} - A_{i, s} \) for the sake of \( R_{5e} \), \( R_{5e+1} \) or \( R_{5e+2} \), we shall also add \( x \) to \( B_{i, s+1} - B_{i, s} \). Notice that changes in \( A_i \) will force changes in the appropriate \( W_i \) or \( M_i \). A requirement \( R_e \) is satisfied, again, if we are preserving a disagreement.

We now describe how \( R_{5e} \) for \( e = (i, j, k) \) requires attention. It does so according to the following options:

(3.2) \( R_e \) has no follower, and
(i) \( s \) is \( i \)-expansionary,
(ii) \( l(i, s) > R((e, j, k), s) \).

(3.3) \( R_e \) has a follower \( x \) such that
(i) \( s \) is \( i \)-expansionary and \( x \) is frozen,
(ii) \( l(e, s) > \max \{u(\Gamma_{i, s}(W_{i, s}; y), u(\Delta_{i, s}(M_{i, s}; y)) : y \leq R(h, s)) \} \).

(3.4) \( R_e \) has an unrealized follower \( x \) such that
(i) \( s \) is \( (i, j) \)-expansionary,
(ii) \( l(i, j, s) > \max \{u(\Sigma_{j, s}(W_{j, s}; y)), u(\Delta_{i, s}(M_{i, s}; y)) : y \leq M \} \), where
(iii) \( M = \max \{u(\Sigma_{j, s}(W_{j, s}; K)), u(\Omega_{j, s}(M_{i, s}; K)), R(h, s) \} \), with \( K \leq K_1 \),
(iv) \( K_1 = u(\Phi_{k, s}(W_{i, s}; x)) \) and
(v) \( \Phi_{k, s}(W_{j, s}; x) \downarrow \).

(3.5) \( R_e \) has a realized follower \( x \), and
(i) \( R_e \) is under attack, and
(ii) \( s \) is \( (e, j) \)-expansionary.

As we remarked earlier the \( R_{5e+j} \) for \( 1 \leq j \leq 4 \), are met by standard methods. For completeness we describe how one of these may require attention. For example let \( j = 1 \). Then \( R_{5e+1} \) requires attention (if unsatisfied) if one option below holds:

(3.6) \( R_{5e+1} \) has no follower.

(3.7) \( R_{5e+1} \) has a follower \( x \) which is unrealized, and \( \Phi_{e, s}(B_{2, s}; x) \downarrow \).

We now describe the construction, again retaining the conventions of Section 2.

**Construction, stage \( s + 1 \)**

For the \( R_{5e+j} \) with \( 1 \leq j \leq 4 \), we appoint an unrealized follower if '3.6' holds and appoint \( (5e + j, Q(s)) \) as a follower. Set \( r(5e + j, s + 1) = (Q(s), Q(s)) \). When '3.7' holds, there are two cases.

Case (i): \( \Phi_{e, s}(:, x) \neq 0 \). Set \( r(5e + j, s + 1) = (Q(s), Q(s)) \).
Case (ii): $\Phi_{e,s}(\cdot;x) = 0$. In this case set the appropriate $T = A_i$ or $B_i$ to be $T_{s+1} = T_s \cup \{x\}$. For example if $j = 1$, set $A_{1,s+1} = A_{1,s} \cup \{x\}$, and $B_{1,s+1} = B_{1,s} \cup \{x\}$, and similarly $j = 2$. If $j = 3$ (or by analogy $j = 4$) set $B_{1,s+1} = B_{1,s} \cup \{x\}$. Set $r(5e + j, s + 1) = \langle Q(s), Q(s) \rangle$.

We now concentrate on the $R_{5e}$.

Case 1: (3.2) holds. Appoint $x = \langle 5e, Q(s) \rangle$ as a follower of $R_{5e}$ which is frozen. Set $r(e, s + 1) = \langle x + 1, x + 1 \rangle$.

Case 2: (3.3) holds, $x$ is no longer frozen and becomes unrealized. Set $T(x) = \langle 5e, Q(s) \rangle$ as the trace of $x$. Set $r(5e, s + 1) = \langle T(x) + 1, T(x) + 1 \rangle$.

Case 3: (3.4) holds, $x$ is now realized. There are two subcases.

Subcase (i): $\Phi_{k,s}(W_j,x) = 1$. Set $r(5e, s + 1) = \langle Q(s), Q(s) \rangle$. $R_{5e}$ is now satisfied.

Subcase (ii): $\Phi_{k,s}(W_j,x) = 0$. Begin an attack as follows: Set $A_{1,s+1} = A_{1,s} \cup \{x\}$, $B_{1,s+1} = B_{1,s} \cup \{x\}$, and set $r(5e, s + 1) = \langle Q(s), Q(s) \rangle$.

Case 4: (3.5) holds. The attack concludes as follows: Set $A_{2,s+1} = A_{2,s} \cup \{x\}$, $B_{2,s+1} = B_{2,s} \cup \{x\}$. Set $V_{i,j,s+1} = V_{i,j,s} \cup \{x\}$, and $r(5e, s + 1) = \langle Q(s), Q(s) \rangle$.

End of Construction

The arguments now are all very similar to those of Section 2. Briefly, to decide if $x \in V_{i,j}$ or not, for the $M_i$ this is permitting. We ask for $t = \mu s \langle M_i[T(x)] = M_{i,s}[T(x)] \rangle$, then $x \in V_{i,j} \leftrightarrow x \in V_{i,j,s+1}$.

Similarly, by delayed permitting, we may find the least stage $s$ where $W_{j,s}[T(x)] = W_j[T(x)]$ (since we can determine if $T(x)$ is defined or not). If no attack has been commenced by stage $x$ and $x \notin A_{1,s}$ then $x \notin V_{i,j}$. If an attack has been commenced, and $x \notin V_{j,s}$, then one is still in progress. Find the least stage $s_i > s$ such that $s_i$ is $(e, j)$-expansionary, or $x$ gets cancelled. Then $x \in V_{i,j} \leftrightarrow x \in V_{i,j,s+1}$.

All of the above again means that each requirement is finitary in nature. Hence they all get met. We leave the reader to complete the details.

End of Construction

Some variations we mention but do not prove are: we may construct $[A_1, B_1] \cap [A_2, B_2]$ not existing in the $W$-degrees. With increased effort as given in Section 2 we may construct $A_i \equiv_T B_i$ for $i, j = 1, 2$. Indeed, it is possible to have $B_2 \triangleleft_T A_1 \triangleleft_T B_1 \triangleleft_T A_2$, and the result still holding. All of these constructions are quite notationally complex, but essentially similar and we leave them to the reader.

It seems conceivable that we might be able to use a strategy similar to ours with perhaps ‘gap co-gap guessing’ to construct $a, b, c$ with $b \wedge c = a$ and $\forall d, e ((b \leq d \leq a \wedge c \leq e \leq a) \rightarrow (d = a \lor a = e \lor d \cap c$ doesn’t exist).

Finally a related question:

(3.8) For $W$-degrees does the ‘generalized non-diamond property’ hold: specifically do there exist $A \upharpoonright_T B$ with $A \cap B$ existing and $A \oplus B = \mu C$ for any given $C$?
4. Pairs with \( \text{inf} \)

We have seen that pairs of r.e. W-degrees may fail to have infimum and yet still, for example, be of the same T-degree. Certainly some pairs of W-degrees do have infimum. In particular, any minimal pair of r.e. T-degrees will give rise to a W-minimal pair. However, we shall explore the situation for comparable T-degrees.

We initially proved our first result directly, but later found a known result (Lachlan [17]) which also gives it, and this is the proof we present.

**Theorem 4.1.** There exist r.e. sets \( \emptyset <_T A \leq_T B \) such that \( A \cap B = \emptyset \). Furthermore,
\[
\forall C ((C \leq_T A \land C \leq_w B) \rightarrow (C = \emptyset)).
\]

**Proof.** In [17], Lachlan embedded the 1-3-1 lattice into the r.e. T-degrees. In particular, there exist r.e. sets \( E, F, G \) such that

1. \( E \oplus F = T E \oplus G = T F \oplus G \),
2. \( E, F, G \) are pairwise T-incomparable,
3. \( e \cap f = e \cap g = f \cap g = \emptyset \).

Let \( A = E \) and \( B = F \oplus G \). Then \( \emptyset <_T A <_T B \). Now let \( D <_T A \) and suppose \( D \leq_w F \oplus G \). By Lachlan’s lemma, \( D = D_1 \sqcup D_2 \) with \( D_1 \leq_w F \) and \( D_2 \leq_w G \). Now, as \( D_1 \leq_w F \) and \( D_1 \leq_T A \), \( D_1 \equiv_T A \). Similarly \( D_2 \equiv_T B \). Hence \( D \equiv_T \emptyset \).

This is a rather pleasing result from the viewpoint of certain questions about splitting properties of r.e. sets and ‘generator’ properties in effective algebra.

Downey and Welch defined an r.e. set \( B \) to have the *antisplitting property* if there exists an r.e. set \( \emptyset <_T A <_T B \), such that if \( B = B_0 \sqcup B_1 \) is an r.e. splitting of \( B \) with \( B_0 \leq_T A \), then \( B_0 \equiv_T \emptyset \). Theorem 4.1 gives a nice example on an r.e. set with the antisolting property.

Also, in effective algebra, must r.e. sets of ‘independent’ generators \( G \) for an r.e. substructure \( V \), have the property that \( G \leq_w V \). Thus, for example, for those familiar with \( L(V_\infty) \), if \( V \in L(V_\infty) \) then:

(4.2) If \( Q \) is an r.e. basis of \( V \), then \( Q \leq_w V \), and

(4.3) If \( P \) is an r.e. subset of a recursive basis of \( V_\infty \), then \( (P)^* \equiv_w P \).

Theorem 4.1 now answers several ‘antibasis’, questions from effective algebra (cf. [10], [27]). For example,

**Theorem 4.4.** There exists an r.e. nonrecursive subspace \( V \) of \( V_\infty \) for which there is an r.e. set \( B \) with \( \emptyset <_T B \leq_T V \) such that if \( Q \) is any r.e. basis of \( V \) with \( Q \leq_T B \), then \( Q \) is recursive.

**Proof.** By (4.1), (4.2) and (4.3) with \( V = (A)^* \) considering \( A \) as an r.e. subset of a recursive basis of \( V_\infty \). \( \square \)
We remark that the analogous result holds in any r.e. structure where 'independent set of generators' is meaningful and (4.2) and (4.3) both hold. Typical examples are boolean algebras, ideals, theories of propositions, free groups, abelian groups and matroids. We direct the reader to [9], [10], [27] and [25] for further details.

It also follows that it would appear to be possible for an r.e. set $A$ to be W-cappable but not T-cappable. However, this is not the case:

**Theorem 4.5.** An r.e. set $A$ has W-cappable W-degree iff it has T-cappable T-degree.

**Proof.** Suppose $A$ is not half of a T-minimal pair. Then, by [5], $A$ has promptly simple degree. Now, in [26, Theorem 1.11] it is shown that then $A$ is not half of a T-minimal pair, by constructing for a given r.e. non-recursive set $B$, an r.e. nonrecursive set $C \leq_T A$, $C \leq_T B$. However, the details of the proof (Lemma 1.7) show that $C \leq_w A$ and $C \leq_w B$, as permitting is used to ensure the reductions. Hence [26] also shows that $A$ is not half of a W-minimal pair. □

W-degrees have often been useful in proving certain properties of the r.e. T-degrees. We feel from this point of view, Theorem 4.1 may have some technical applications. For example it would appear to be somewhat easier to construct an r.e. non-zero degree W-bounding no T-comparable W-minimal pair than to construct a Lachlan nonbounding degree [18], [32]. It would therefore be interesting to know the answer to the question:

(4.6) Let $a$ be an r.e. degree such that $a \neq 0$ and $\forall b (0 <_T b \leq_T a) \rightarrow b$ bounds a minimal pair. Does $a$ bound a 1-3-1 lattice, or (at least) a T-comparable W-minimal pair?

A positive solution to (4.6) could possibly simplify many of the proofs about (non) bounding minimal pairs. A negative solution would give new 1-types in the r.e. T-degrees differing from those of Ambos-Spies and Soare [6]. From [6] we do have the following:

**Theorem 4.7.** There exists an r.e. degree $a$ such that $a$ bounds no 1-3-1 lattice (in fact, no T-comparable W-minimal pair) but $a$ bounds a minimal 'triple'. Namely there exist $c, d$ and $e$ with $0 <_T c, d, e <_T a$, and $c, d$ and $e$ are pairwise minimal pairs.

**Proof.** In [6], [cf. [2]], r.e. nonrecursive sets $C, D, E$ are constructed such that

(i) $c \cap d = d \cap e = c \cap e = 0$,
(ii) $c, d, e, c \oplus d, d \oplus e, e \oplus e$, and $c \oplus d \oplus e$, are all contiguous, and
(iii) $c, d, e$ each bound no minimal pairs
(iv) $(c \oplus d) \cap e = c \cap (d \oplus e) = d \cap (c \oplus e) = 0$. 
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Let $\emptyset \triangleleft_T B \triangleleft_T A$ with $A \leq_T C \oplus D \oplus E$. By contiguity $A <w C \oplus D \oplus E$ and $B <w C \oplus D \oplus E$. Thus, by Lachlan's lemma,

$$A = A_1 \sqcup A_2 \sqcup A_3 \quad \text{and} \quad B = B_1 \sqcup B_2 \sqcup B_3$$

with

$$A_1, B_1 \leq_w C; \quad A_2, B_2 \leq_w D; \quad \text{and} \quad A_3, B_3 \leq_w E.$$ 

Now suppose $A \cap B = \emptyset$. By (iii), for all $i$, if $A_i \not\equiv_T \emptyset$, then $B_i \equiv_T \emptyset$ and if $B_i \not\equiv_T \emptyset$ then $A_i \equiv_T \emptyset$. As $B \not\equiv_T \emptyset$, we may suppose $B_1 \not\equiv_T \emptyset$. Hence $A_1 \equiv_T \emptyset$. Consequently, $A \leq_w D \oplus E$. Now $B \leq_T A$. Hence $B \leq_T D \oplus E$. By contiguity, $B <_w D \oplus E$. Hence $B_1 \leq_w D \oplus E$. By (iv) as $B_1 \leq_w C$ and $B_1 <_w D \oplus E$, it follows that $B_1 \equiv \emptyset$, a contradiction, giving the result. \(\square\)

We return to the structure of W-degrees within a T-degree. We have seen that in a very strong way the 'W-inf' of a pair of r.e. W-degrees of the same T-degree may be forced out of the degree. We show, extending (4.1) in this sense.

**Theorem 4.8.** There exist r.e. sets $A \equiv_T B \not\equiv_T \emptyset$ such that $A \cap B = \emptyset$.

**Remark.** We could obtain this as in (4.1) using the 1-4-1 lattice. We prefer a direct proof for later applications.

**Proof.** We shall build $A = \bigcup_e A_e$ and $B = \bigcup_e B_e$ and satisfy the following requirements:

$P_e$: $\bar{A} \not= W_e$,

$N_e$: $\Phi_e(A) = \Phi_e(B) = f$ and $f$ total $\rightarrow f$ recursive,

and, in addition, arrange that $A \leq_T B$ and $B \leq_T A$.

Our construction will be by the pinball machine model. We shall briefly discuss why we choose this model. For one requirement, our idea is the usual minimal pair one, combined with a tracing procedure.

For example, we may wish to add a follower $x$ to $A$ for the sake of $P_e$. For the sake of $N_e$ we shall do so at an $e$-expansion stage (as usual), but only after we've traced $x$ in $B$. For example, we might have the situation (at stage $s$)

$$\begin{array}{cccccccc}
& & x & \phi_{e,s} & A_s \\
& & \phi_{e,s} & & \\
& f_s & & y & \phi_{e,s} & B_s \\
\end{array}$$

We wish to add at stage $s$, a follower $x$ to $A$ (as above) and to ensure that $A \leq_T B$ also add a trace $y$ into $B$. However, it may be that both $x$ and $y$ are less than the restraint imposed by the minimal pair strategy, which requires us to restrain at least one side of the computations, $\Phi_{e,s}(A_s) = \Phi_{e,s}(B_s) = f_s$ at initial segment of $\omega$. 


What we do is then add $y$ to $B$ (this won't injure the minimal pair strategy) and choose a new trace $y'$ for $x$. The key is that $y'$ may be chosen larger than the use of the computations at stage $s$, and since we are dealing with $W$-computations this use function is fixed and in particular must be less than $y'$. In the actual construction, things are a little more complicated, since we also need $B \preceq_T A$, and hence we shall need a long series of descendents of $x$ (traces, traces of traces of traces, etc.) as in Theorem 2.2, but again this can be overcome in a similar method to that used in (2.2). The principal difficulty in the construction is in the interaction between tracing and the co-ordination of the requirements. For example, if we attempt to use a tree-style argument, instead of $e$-expansionary stages, we will have $\sigma$-expansionary stages. (We here assume the reader is familiar with this argument.) Thus after adding $x$ to $B$, we must wait for the next $\sigma$-expansionary stage. Now, it may very well be the case that at this stage $t$, the following situation occurs.

Now, although we can add $x$ to $A_{t+1} - A_t$ and $y'$ to $B_{t+1} - B_t$, and this will not change $f$ it is now possible that it could change $f$, since both $x$ and $y'$ are within the critical zone corresponding to the uses of $f$.

Our solution is to use a pinball machine which allows us to eventually act soon enough to avoid this situation. The point being that in pinball constructions we rely only on $e$- rather than $\sigma$-expansionary stages. We remark that similar problems occur in constructions embedding nondistributive lattices into the r.e. $T$-degrees. (It was only after noticing these similarities, that we discovered the proof of Theorem 4.1, using Lachlan’s [17] theorem.)

The combinatorial picture of this proof is the pinball machine below. This
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model, together with the associated terminology is due to Lerman [23].

Hole $H_e$ is assigned to $P_e$, and Gate $G_i$ on $N_e$. On the surface of the machine at any stage, are balls marked $\langle x, e, A \rangle$ or $\langle x, e, y, C \rangle$ where $C = A$ or $B$. Balls marked $\langle x, e, A \rangle$ are followers, and will be emitted by $H_e$. Balls marked $\langle x, e, y, C \rangle$ are trace balls, and will be descendents of some follower. Such a ball is attempting to enumerate $x$ into $C = A(B)$ as a trace for $y$ in $B(A)$. If $\langle x, e, y, C \rangle$ is a trace ball, there is also a ball $\langle y, c, A \rangle$ or $\langle y, c, z, D \rangle$ with $\{C, D\} = \{A, B\}$ on the surface. Balls are either on the surface or in the pockets. A ball on the surface may be at a gate, hole, or in a corral. Finitely many balls may be at a gate or hole at any time, but if there is more than one, they have the following relationship with one another:

$$\langle (x, e, A) \rangle, \langle y_1, e, x, B \rangle, \langle y_2, e, y_1, A \rangle, \ldots, \langle y_n, e, y_{n-1}, C \rangle$$

with $C = A$ or $B$. That is, a follower or its descendents. Trace balls have the same priority as the balls they descend from. A follower $\langle x, e, A \rangle$ has higher priority than a follower $\langle y, f, A \rangle$ iff $e < f$ or $e = f$ and $x < y$.

As in our Section 2 construction, all trace balls must get enumerated before a follower gets enumerated. Let

$$l(e, s) = \max \{x : \forall y < x [\Phi_{e,s}(A; y) = \Phi_{e,s}(B; y)]\},$$

and

$$ml(e, s) = \max \{l(e, t) : t < s\}.$$ 

$P_e$ requires attention at stage $s + 1$ if

(4.9) There is a gate $G_i : i \leq e$ and a ball $\langle x, e, A \rangle$ or $\langle x, e, y, C \rangle$ with $C = A$ or $B$ at gate $G_i$ and $l(i, s) > ml(i, s)$, or

(4.10) There is a ball $\langle x, e, A \rangle$ above $H_e$ such that $x \in W_{e,s}$, or

(4.11) $W_{e,s} \cap A_e = \emptyset$ and no balls are above $H_e$.

Construction, stage $s + 1$

Step 1. Find the least $e$ such that $P_e$ requires attention. Discard all balls of the form $\langle y, i, A \rangle$ or $\langle y, i, y', C \rangle$ for $i > e$. Adopt the first case which pertains to $e$.

Case 1: (4.9) holds. Find the ball of highest priority and the corresponding $i$. We claim.

Claim (a). The balls at $G_i$ will form a sequence $\langle x, e, , C \rangle$, $\langle y_1, e, x, D \rangle, \ldots, \langle y_n, e, y_{n-1}, C(D) \rangle$ where $C, D = A$ or $B$. Discard all balls of lower priority than these. Now, let $j < i$ be the largest $j$ (if any) with $G_j$ unoccupied. Release all the balls above from gate $G_i$, place $\langle y_n, e, y_{n-1}, C(D) \rangle$ at gate $G_j$ and the remaining balls in corral $C_j$. If $j$ does not exist, put each ball into the appropriate pocket. In addition, if the first ball $\langle x, e, , C \rangle$ is a trace ball, say $\langle x, e, z, C \rangle$, find the ball marked $\langle z, e, x', D \rangle$. We claim
Claim (b). \( \langle z, e, x', D \rangle \) is in a corral \( C_k \) for \( e \geq k \geq i \), and

Claim (c). \( G_k \) is empty.

Place \( \langle z, e, x', D \rangle \) at gate \( G_k \).

Case 2: (4.10) holds. We claim

Claim (d): The balls above \( H_e \) will form a sequence \( \langle x, e, A \rangle, \langle y_1, e, x, B \rangle, \ldots, \langle y_n, e, y_{n-1}, C \rangle \).

Proceed as in Case 1, replacing \( i \) by \( e + 1 \).

Case 3: (4.11) holds. Let \( x = Q(s) + 1 \). Mark a ball \( \langle x, e, A \rangle \) and place it above hole \( H_e \).

Step 2 (Appointing traces). For each ball on the surface of the machine which does not already have a trace on the surface (that is, it never had a trace ball, or its trace ball previously appointed is now in a pocket), appoint a trace in the obvious way. Place the trace ball in the same place as the original ball.

Claim (e): This will necessarily be at a gate or at a hole. \( \square \) End of Construction.

Lemma 4.12. (i) Suppose \( \langle y_0, e, A \rangle \) is a follower of \( P_e \) on the surface at stage \( s \), and \( \langle y_1, e, y_0, B \rangle, \ldots, \langle y_n, e, y_{n-1}, A(B) \rangle \) is its sequence of traces on the surface. Then either they are all at \( H_e \), or at some gate \( G_i \) for \( i \leq e \), or there is a sequence \( i_k \leq i_{k-1} < \cdots < i_1 \leq e \), and a sequence \( 0 \leq j_1 < j_2 < \cdots < j_{k-1} < n \), such that

\[
\langle y_0, e, A \rangle, \langle y_1, e, y_0, B \rangle, \ldots, \langle y_{j_1}, e, y_{j_1-1}, C \rangle \text{ are at corral } C_{i_1},
\]
\[
\langle y_{j_1+1}, e, y_{j_1}, D \rangle, \ldots, \langle y_{j_2}, e, y_{j_2-1}, C(D) \rangle \text{ are at corral } C_{i_2},
\]
\[
\cdots
\]
\[
\langle y_{j_{k-2}+1}, e, y_{j_{k-2}}, C(D) \rangle, \ldots, \langle y_{j_{k-1}}, e, y_{j_{k-1}-1}, C(D) \rangle \text{ are at corral } C_{i_{k-1}},
\]
\[
\langle y_{j_{k-1}+1}, e, y_{j_{k-1}}, C(D) \rangle, \ldots, \langle y_n, e, y_{n-1}, C(D) \rangle \text{ are at gate } G_{i_k}.
\]

(ii) Furthermore, Claims (a)–(e) made in the construction all hold at each stage \( s \).

Proof. By induction. \( \square \)

Lemma 4.13. For all \( e \), \( P_e \) is satisfied and receives attention at most finitely often.

Proof. Suppose not. By induction, suppose that \( P_j \) for \( j < e \) don't receive attention after stage \( s_0 \). If \( W_{e,s} \cap A_s \neq \emptyset \) for any \( s > s_0 \), we are home. So suppose \( W_e \cap A = \emptyset \).

(4.14) A single follower can be responsible for \( P_e \) receiving attention at most finitely often.

To see this, let \( \langle x, e, A \rangle \) be the follower of highest priority responsible for \( P_e \) receiving attention infinitely often. By (4.12) at any stage \( s \), the balls associated
with $x$ must lie in the given configuration. Now, at any stage $s$, if the balls are in the third configuration and some ball associated with $\langle x, e, A \rangle$ receives attention, then it can only be that for some $i_k$, the balls at gate $G_{i_k}$ are released, drop to a new unoccupied gate, and one ball from the corral $C_{i_k}$, is released (that is, placed at the gate $G_i$ which is unoccupied). We can see that by induction, if the balls associated with $\langle x, e, A \rangle$ receive attention often enough, then eventually $\langle x, e, A \rangle$ must be enumerated into meeting $P_e$.

We now argue that the $P_e$ are met. Let $s_1$ be the least stage such that balls associated with $\langle x, e, A \rangle$ receive attention for the last time. If $\langle x, e, A \rangle$ is above hole $H_e$, then $P_e$ is met since this is an unrealized follower. Otherwise some sequence of balls given by Lemma 4.12 are at some gate $G_{i_k}$ permanently. Now, assuming $P_e$ fails to be met, or receives attention infinitely often, after stage $s_1$ (say at stage $s_2$) $P_e$ will get a new follower $\langle x_2, e, A \rangle$. Arguing similarly, balls associated with $x_2$ must come to rest at some gate $G_{i_k}$. However $i_k \neq i_n$. We cannot place balls associated with $\langle x_2, e, A \rangle$ at $G_{i_k}$ since (higher priority) balls associated with $\langle x, e, A \rangle$ are already (permanently) there, and balls drop to unoccupied gates. It is easy to see that the $(e + 1)$th follower must eventually enter a pocket, since all the gates will be occupied.


Proof. Any ball targeted for $A$ is either $\langle x, e, A \rangle$ or an $\langle x, e, y, A \rangle$ ball. By the trace assignment step, it must get a trace, or is cancelled. Find the stage $s$ where either $x$ gets cancelled, $s > x$ (so it is never appointed as a trace or follower), $x$ gets a trace $\langle y, e, x, B \rangle$, or $x \in A$. Assuming $x$ gets a trace $y$, ask if $y \in B$. If $y \notin B$, then $x \notin A$. If $y \in B$ find the stage $s_1$ where $y \in B_{s_1} - B_{s_1 - 1}$. If $x \notin A_{s_1}$, then $x$ gets a new trace at stage $s_1$. In fact, by the construction, $x$ is also removed from the corral it was in and placed at gate $G_i$, and then given a new trace, $\langle y_1, e, x, B \rangle$. If $y_1 \notin B$, then $x \notin A$. If $y_1 \in B$, then at some stage $s_2 > s_1$, both $x$ and $y_1$ pass gate $G_i$. (Both balls get released at the same stage, and must be released if $y_1 \in B$.) The point is, when $y \in B_{s_2} - B_{s_2 - 1}$ for $s_2 \geq s_1$, if $x \notin A_{s_2}$, then it must have been in a corral $C_j$ but $j < i$. It follows that $x$ can be given at most $e$ new traces. Thus $B$ can determine whether or not $x \in A$, and so $A \equiv_T B$. That $B \equiv_T A$ is similar.

Lemma 4.15. All the $N_e$ are met.

Proof. Let $\Phi_e(A) = \Phi_e(B) = f$ with $f$ total. We show how to compute $f(p)$. First compute $\phi_e(p) = q$, say. Let $s_0$ be a stage such that

1. $P_i$ for $i < e$ don't receive attention after $s_0$.
2. all residents of $G_i$ and $C_i$ for $i \leq e$ at stage $s_0$ are permanent residents,
3. $l(e, s_0) > p$,
4. $s_0 \geq q$.

We claim (a) $s_0$ exists, and (b) $s_0$ may be found effectively.
For claim (a) we see that (1), (3) and (4) happen cofinitely often. Let \( t_0 \) be the least stage at which (1), (3) and (4) occur. Set \( t_1 > t_0 \) to be the least stage at which (1), (3) and (4) occur. Set \( t_1 > t_0 \) to be the least stage at which the ball of highest priority ever to act after stage \( t_0 \), acts for the last time. Then every ball remaining on the surface is in its final position.

Now for claim (b), although gates may get infinitely many permanent residents, they will all be descendents of some initial permanent resident. Let \( s \) be the least stage such that every gate \( G_i \), with \( i < e \), which receives a permanent resident, has one by stage \( s \) and \( s > t_0 \) (where \( t_0 \) as above); and every corral \( C_i \), with \( i < e \), that gets permanent residents has them at stage \( s \). It is easy to see that we can check (1)–(4) against this list of parameters, to effectively find \( s_0 \). (Of course, we don't claim to be able to find \( t_1 \), as above.)

We claim:

\begin{equation}
(4.16) \quad f(p) = \Phi_{e,s_0}(A_{s_0}; p) = \Phi_{e,s_0}(B_{s_0}; p).
\end{equation}

Let \( m = \Phi_{e,s_0}(A_{s_0}; p) \). We shall show that (4.16) holds by proving

\begin{equation}
(4.17) \quad \forall t > s_0 \ (\Phi_{e,t}(A_i; p) = m \text{ or } \Phi_{e,t}(B_i; p) = m).
\end{equation}

Otherwise, let \( s_1 + 1 > s_0 \) be the first stage at which (4.17) fails. Then at stage \( s_1 + 1 \), some ball must have entered \( A \) or \( B \) (or both), and must be marked \( \langle x, \ldots \rangle \) with \( x < q \). (Otherwise, as (4.16) was true at \( t = s_1 \), it is also true at \( s_1 + 1 \).) Thus let \( \langle x, e, C \rangle \) be the ball with \( x \) least entering \( A \) or \( B \) at stage \( s_1 + 1 \), \( x < q \). Since \( x < q < s_0 \), it follows that \( \langle x, i, C \rangle \) must have been on the surface of the machine at stage \( s_0 \), and furthermore, have been above gate \( G_e \). Thus there is a stage \( s_2 + 1 \) such that at stage \( s_2 + 1 \), \( \langle x, \ldots \rangle \) either

\begin{equation}
(4.18) \quad \text{passes gate } G_e, \text{ or}
\end{equation}

\begin{equation}
(4.19) \quad \text{leaves gate } G_e,
\end{equation}

and \( s_0 < s_2 + 1 < s_1 + 1 \).

Case 1: (4.18) holds. This can only occur if some ball \( y \) of higher priority is already at gate \( G_e \). It must have come to rest there at some stage \( s_3 \) with \( s_0 < s_3 < s_2 + 1 \). But at this stage \( s_3 \), \( y \) would have cancelled \( x \), and so (4.18) cannot hold.

Case 2: (4.19) holds. \( x \) leaves \( G_e \) at \( s_2 + 1 \). Now, \( x \) was placed at gate \( G_e \) at stage \( s_3 + 1 \) with \( s_0 < s_3 + 1 < s_2 + 1 \). At \( s_2 \), \( l(e, s_2) > ml(e, s_2 - 1) \geq l(e, s_0) \geq p \). Thus we know

\begin{equation}
(4.20) \quad \Phi_{e,s_2}(A_{s_2}; p) = m = \Phi_{e,s_2}(B_{s_2}; p).
\end{equation}

Now, if \( t \) is any stage with \( s_2 + 1 < t < s_1 + 1 \) and a ball \( y \) is placed in a pocket at stage \( t \), that ball is not of higher priority than \( x \) (otherwise \( y \) would cancel \( x \) at stage \( t \)). Consequently, any such \( y \) must have been appointed after stage \( s_2 + 1 \) (since otherwise it would cancel \( x \)). Consequently, \( y > \Phi(p) \).

It thus follows that the only balls that can destroy computations by stage \( s_1 + 1 \)
must be of the same priority as \( x \). But at stage \( s_1 + 1 \), only \( x \) and its descendents (or predecessors) can be such balls. By the construction at stage \( s_3 + 1 \), \( x \) either came out of corral \( C_e \), or \( x \) has no predecessors not in corrals, (and these must wait for \( x \) to enter \( A \)). In either case, \( x \) is appointed a trace \( T \) at stage \( s_3 \) with \( T > s_3 \). It therefore follows that no number \( y \leq \phi(p) \) with \( x \neq y \) may enter \( A \) or \( B \) at any stage \( t \) with \( s_2 + 1 \leq t \leq s_3 + 1 \). Thus one computation must still hold at stage \( s_1 + 1 \), hence (4.17) and thus (4.18) follows. \( \Box \)

**Corollary 4.21.** There exists an r.e. degree \( a \neq 0 \) such that for all r.e. \( B \) with \( b \leq_T a \), there exist r.e. sets \( A_1, A_2 \) with \( A_1, A_2 \in a \) and \( A_1 \cap A_2 = B \).

**Proof.** Let \( A =_T B \) be given by Theorem 4.8. Let \( C \) be r.e. with \( c \leq a \). Now let \( A_1 = C \oplus A \) and \( A_2 = C \oplus B \). Now \( C \leq a_1, a_2 \). Now let \( E \leq_w a_1, a_2 \). By Lachlan’s lemma, \( E = E_1 \uplus E_2 \) with \( E_1 \leq_w C \) and \( E_2 \leq_w A \). Similarly, \( E_2 = F_1 \uplus F_2 \) with \( F_1 \leq_w C \) and \( F_2 \leq_w B \). Now as \( F_2 \leq_w A \), \( B, F_2 =_T \emptyset \). Hence \( E \leq_w C \). \( \Box \)

We remark that Theorem 4.8 implies Theorem 4.1 as follows. Let \( A =_T B \) be given by (4.8). By Ladner and Sasso [22], let \( B' \) be a contiguous r.e. nonrecursive set with \( B' \leq_w B \). Then \( B' \cap A = \emptyset \) since \( B \cap A = \emptyset \), and for all \( C \leq_T B', C \leq_w B' \) by contiguity.

We may extend Theorem 4.8 as follows in Theorem 4.22. We remark that we do not know if there is an r.e. degree \( a \) such that for all \( A \in a \), there is a \( B \in a \) with \( A \cap B = \emptyset \). We have

**Theorem 4.22.** There exists an r.e. set \( A \) such that for all r.e. \( W \leq_T A \), there is an r.e. \( B \leq_T A \) with \( B \neq_T \emptyset \) and \( W \cap B = \emptyset \).

**Proof.** We build \( A = \bigcup_{s} A_s \) and \( B_e = \bigcup_{s} B_{e,s} \) satisfying

\[
\begin{align*}
P_{(e,j)}: & \text{ If } \Psi_e(A) = W_e, \text{ then } B_e \leq_T A \text{ and } B_e \neq W_e, \\
N_{(e,j)}: & \text{ If } \Psi_e(A) = W_e, \text{ then if } f(W_e) = f(B_e) = f \text{ and } f \text{ is total then } f \text{ is recursive.}
\end{align*}
\]

Again, our construction is by pinball machine. This time, however there are no corrals.

---

**Diagram:**

- Hole \( H_{(e,j)} \)
- Gate \( G_{(e,j)} \)
- Disc \( A, B_0, B_1, \cdots \)
Hole $H_{(e,j)}$ represents $P_{(e,j)}$ and gate $C_{(e,j)}$ represents $N_{(e,j)}$. There are infinitely many pockets at the bottom of the machine. A gate $G_{(h,k)}$ is called an $e$-gate if $h = e$. Again $(\cdot, \cdot)$ is monotone in the second variable, so $G_{(e,j)}$ will be below $G_{(e,k)}$ if $j < k$. A ball on the surface may be active, waiting or frozen. Freezing indicates that it is waiting for a trace. If $\Psi_e(A) = W_e$, any ball which becomes frozen will be activated or cancelled. At any stage $s$, a ball has at most one trace on the surface. The general progress of a follower ball is active-waiting-frozen-active etc. Traces are always waiting. Let

$$L(e, s) = \max \{ \forall y \leq x [\Psi_{e,s}(A; y) = W_{e,s}(y)] \},$$

$$mL(e, s) = \max \{ L(e, t) : t < s \}$$

and let

$$l(e, j, s) = \max \{ \forall y \leq x [\Phi_{j,s}(W_{e,s}; y) = F_{j,s}(B_{e,s}; y)$$

$$\& \forall z (z \leq \max \{ \phi_{j,s}(y), \gamma_{j,s}(y) \} \rightarrow (\Psi_{e,s}(A; z) = W_{e,s}(z)) \} \},$$

and finally let

$$ml(e, j, s) = \max \{ l(e, j, t) : t < s \}.$$ 

Priority of balls is as in (4.8). Followers of $P_{(e,j)}$ will be marked $x(e, j)$ and traces will be marked $y(e, j, x)$. We shall say $P_{(e,j)}$ requires attention at stage $s + 1$, if $B_{e,s} \cap W_{j,s} = \emptyset$ and

(4.23) A follower $x(e, j)$ is at gate $G_{(h,k)}$ and
(i) $x(e, j)$ is active,
(ii) $l(h, k, s) > ml(h, k, s),$

(4.24) A trace $y(e, j, s)$ is at gate $G_{(h,k)}$ and
(i) $l(h, k, s) > ml(h, k, s),$
(ii) $x(e, j)$ is waiting,

(4.25) A follower $x(e, j)$ is at gate $G_{(e,k)}$ and
(i) $x(e, j)$ is frozen,
(ii) $L(e, s) > mL(e, s),$

(4.26) A follower $x(e, j)$ with trace $y(e, j, x)$ is above hole $H_{(e,j)}$ and $x \in W_{j,s},$

(4.27) There is no follower above hole $H_{(e,j)}$.

**Construction, stage $s + 1**

**Step 1.** Find the ball of highest priority requiring attention. Cancel all balls of lower priority.

**Step 2.** Adopt the appropriate case.

**Case 1:** (4.23) holds. Release $x(e, j)$ and its trace $y(e, j, x)$ from gate $G_{(h,k)}$ and place them at the first gate below $G_{(h,k)}$ unoccupied save for frozen followers. If
Recursively enumerable T- and W-degrees

$G_{(h,k)}$ is not an $e$-gate, then $x(e, j)$ remains active. If $G_{(h,k)}$ is an $e$-gate (recall, then $G_{(e,k)} = G_{(h,k)}$), declare $x$ as waiting.

If no such gate exists enumerate $x$ into the $B_e$-pocket and $y$ into the $A$-pocket and cancel all followers, traces of $P_{(e,i)}$.

Case 2: (4.24) holds. Allow $y(e, j, x)$ to drop to the first gate unoccupied save for frozen followers. If such a gate exists, $x(e, j)$ remains waiting. If no such gate exists, enumerate $y$ into the $A$-pocket and declare $x(e, j)$ as frozen.

Case 3: (4.25) holds. $x(e, j)$ becomes active. Appoint $y(e, j, x) = s + 1$ as a trace for $x$ (targeted for $A$).

Case 4: (4.26) holds. Place $x(e, j)$ and $y(e, j, x)$ at gate $G_{(e,i)}$ and declare $x(e, j)$ as waiting.

Case 5: (4.27) holds. Appoint $x(e, j) = y(e, j, x) = s + 1$ as a follower and trace of $P_{(e,i)}$ respectively and place them above hole $H_{(e,i)}$. \[\square\] End of Construction

**Verification.** It is easily established by induction that all the $P_{(e,i)}$ require attention at most finitely often and are met. Similarly $B_e \leq_T A$ for all $e$ by the tracing procedure (if $\Psi_e(A) = W_e$). These are similar to Theorem 4.8 (but easier).

The key to the proof is thus:

(4.28) If $\Psi_e(A) = W_e$, then if $\hat{\Phi}_{j}(W_e) = \hat{\Gamma}_{j}(B_e) = f$ with $f$ total, then $f$ is recursive.

We prove this. Let $p$ be given. We can, using an argument similar to (4.8), find a stage $s_0$ such that for all $t > s_0$, for all $k \leq (e, j)$,

(i) $P_k$ doesn’t receive attention at stage $t$,

(ii) $G_k$ has only permanent residents at stage $s_0$,

(iii) if $G_k$ doesn’t get permanent residents, it has no residents at stage $s_0$,

(iv) $l(e, j, s) > p$,

(v) $s_0 > u(\hat{\Gamma}_{j,s_0}(B_{e,s_0}; p)) = \gamma_{j,s_0}(p)$.

We claim $\forall s > s_0 ((\hat{\Phi}_{j,s}(W_{e,s}; p) = f_{s_0}(p)) \lor (\hat{\Gamma}_{j,s}(B_{e,s}; p) = f_{s_0}(p)))$. Let $s_1 + 1$ be the first counterexample. Let

\[u(s) = u(\Psi_{e,s}(W_{e,s}; \phi_{j,s}(p))).\]

Now, we know that between stages $s_0$ and $s_1 + 1$, some number $\leq u(s_0)$ must enter $A$ and some number $\leq \gamma_{j,s}(p)$ must enter $B_e$. Let $x$ be the first number $\leq \gamma_{j,s}(p)$ to enter $B_e$ say at stage $s_2$ with $s_0 < s + 1 \leq s_1 + 1$. It follows that $x$ is a follower of some $P_{(e,k)}$ and $x$ was already on the surface at stage $s_0$. Arguing by priorities, both $x$ and its trace $y$ must stop at gate $G_{(e,i)}$ say at stage $s_3$, with $s_0 < s_3 < s_2 + 1$. At such a stage $x$ will be frozen. Since $x$ enters $B_e$, it becomes ‘thawed’ at some stage $s_4$, with $s_3 < s_4 < s_2 + 1$, and appointed a trace $y'$ with $y' > u(s_4)$. Furthermore $(x, y')$ must be released at some stage $s_5 + 1$, with $s_4 < s_5 + 1 \leq s_2 + 1$, so that at stage $s_5$, $l(e, j, s) > p$. (In particular, at stage $s_5$, $\hat{\Phi}_{j,s_5}(W_{e,s_5}; p) = f_{s_0}(p) = \hat{\Gamma}_{j,s_5}(B_{e,s_5}; p).)$
Now, arguing by priorities, at stage $s_2$, since $x$ has not been cancelled (after all $x \in B_{e,s_2+1} - B_{e,s_2}$),

(i) $\Phi_{j,s_2}(W_{e,s_2}; p) = f_{s_2}(p) = \hat{f}_{j,s_0}(B_{e,s_0}; p)$, and

(ii) for all $z$, if $z \leq u(s_2)$ and $z$ is on the surface, then $z$ is above gate $G_{e,j}$, since $z$ cannot have moved in all this time (lest they cancel $x$), or $z$ follows the same requirement as does $x$.

Now, at stage $s_2 + 1$ all lower priority followers are cancelled, and all followers of the same requirement are also cancelled, it follows that

(i) $\Phi_{j,s_2+1}(W_{e,s_2+1}; p) = f_{s_2}(p)$, and

(ii) $\forall z \leq u(s_2 + 1) (z$ is above gate $G_{e,j}$).

Now, the computations may only be upset if some number enters $W_{e,s_2+1} - W_{e,s_2}$ with $z < u(s_2 + 1)$. Any such number must be on the surface at stage $s_2 + 1$, and hence above gate $G_{e,j}$. Arguing by priorities, $z$ must stop at gate $G_{e,j}$ as it is not cancelled. But now $z$ cannot leave gate $G_{e,j}$ until $l(e, j, s) > ml(e, j, s_2 + 1)$. Hence at the stage $s_0 + 1$ when $z$ leaves gate $G_{e,j}$ we know $\Phi_{j,s_0}(W_{e,s_0}; p) = f_{s_0}(p) = \hat{f}_{j,s_0}(B_{e,s_0}(p))$.

In this way, we see that at all stages $t > s_0$ at least one computation must hold, giving (4.18) and the theorem.

**Corollary 4.29.** There exists an r.e. degree $\delta \neq 0$ such that

(i) for all r.e. sets $A \in \delta$, $A$ has the antisplitting property, and

(ii) for all $V \in L(V_\omega)$, if $\deg(V) = \delta$ then $V$ has the antibasis property.

**Proof.** (for example (ii)). Let $V \in L(V_\omega)$ with $V$ given by Theorem 4.22 of degree $A$. Let $B$ be a contiguous r.e. set with $B \cap V = \emptyset$. Let $R$ be an r.e. basis of $V$ with $R = T B$. Then, by contiguity, $R \leq T B$ and so $R = T \emptyset$. Hence $V$ has the antibasis property. 

The pinball techniques we have described, have several other useful applications. For our final result, we shall use them to solve a question from [11]. So far, every r.e. set to be constructed with the antisplitting property has cappable degree. It was thought to be possible that the converse also held. Viz, in general, if $A$ has the antisplitting property, then $a$ is cappable. However, we shall show how to construct an r.e. set of promptly simple degree with the antisplitting property. Indeed

**Theorem 4.30.** There exists a complete r.e. set with the antisplitting property.

**Proof.** Let $f$ be a 1-1 recursive function with $f(\omega) = T \emptyset'$. We shall build $A = \bigcup_s A_s$ and ensure $A = T f(\omega)$ by marker coding. Thus, at each stage $s$, \{a_{i,s}: i \in \omega\} list in order $\omega - A_s$. We promise to add $a_{f(\omega),s}$ to $A_{s+1} - A_s$ for all $s$. This will suffice.
We shall also build \( B = \bigcup_s B_s \) and satisfy

\[ P_e: \quad B \neq W_e \text{ (by followers)}, \]

\[ N_e: \quad \text{If } W_e \cap V_e = A \text{ and } \Phi_e(B) = W_e, \text{ then } W_e \text{ is recursive.} \]

Our construction is by pinball machine with gates and holes arranged in the usual way. Associated with \( P_j \) will be a restraint \( r(j, s) \) which is defined as follows: Let

\[ l(e, s) = \max \{ x : \forall y < x((W_{e,s} \sqcup V_{e,s})[y] = A_s[y] \& \Phi_{e,s}(B_s; y) = W_{e,s}(y)) \}, \]

and let \( ml(e, s), u(e, x, s) \) be defined similarly. Then

\[ r(e, j, s) = \begin{cases} 
\max \{ y : a_{j,s} \leq y ; u(\Phi_{e,s}(B_s; y)) \} & \text{if } l(e, s) > a_{j,s}, \\
\max \{ r(e, j, s) : e \leq j \} & \text{otherwise},
\end{cases} \]

For any follower \( x \) of \( P_j \) if, at any stage \( s \) for any \( k \leq j, r(k, s) \geq x \) and \( x \) is on the surface we cancel \( x \). Also, at any stage \( s \) if \( f(s) \leq j \), we cancel all followers of \( P_j \) on the surface.

We say \( P_e \) requires attention at stage \( s + 1 \) if \( B_s \cap W_{e,s} = \emptyset \) and one of the following options hold:

\[ (4.31) \quad \text{There is a follower } x \text{ of } P_e \text{ at gate } G_j \text{ and } l(j, s) > ml(j, s). \]

\[ (4.32) \quad \text{There is a follower } x \text{ of } P_e \text{ above hole } H_e \text{ and } x \in W_{e,s}. \]

\[ (4.33) \quad \text{There is no follower of } P_e \text{ above hole } H_e. \]

**Construction, stage \( s + 1 \)**

**Step 1.** For all \( j \) with \( f(s) \leq j \), cancel all balls associated with \( P_j \). For all balls \( x \) associated with \( P_k \) and \( x \leq r(j, s) \) for some \( j \leq k \), cancel \( x \).

**Step 2.** Find the requirement \( P_e \) which requires attention. Cancel all balls of lower priority. Adopt the appropriate case below.

**Case 1:** (4.31) holds. Allow \( x \) to drop to the first unoccupied gate. If no such gate exists, set \( B_{s+1} = B_s \cup \{ x \} \) and \( A_{s+1} = A_s \cup \{ a_{f(s),s} \} \). If such a gate exists, set \( A_{s+1} = A_s \cup \{ a_{f(s),s}, a_{e,s}, \ldots, a_{s,s} \} \) and \( B_{s+1} = B_s \). (Notice as \( x \) is still on the surface, \( f(s) > e \).)

**Case 2:** (4.32) holds. Same as Case 1.

**Case 3:** (4.33) holds. Appoint \( x = s \) as a follower of \( P_e \). Move it above hole \( H_e \). Set \( A_{s+1} = A_s \cup \{ a_{f(s),s} \} \) and \( B_{s+1} = B_s \). \( \square \) End of Construction

**Lemma 4.34.** \( a = 0' \), \( \lim_s a_{i,s} = a_i \) exists and all the \( P_e \) receive attention at most finitely often and are satisfied, and \( \lim_s r(i, s) = r(i) \) exists.
Proof. Let $t_1$ be the least stage such that, by induction, for all $j \leq e$, for all $s > t_1$

(i) $a_{j,s} = a_j = a_{j,t_1},$
(ii) $r(j, s) = r(j) = r(j, t_1).$
(iii) $f(s) > e + 1,$
(iv) $P_k$ doesn't receive attention at stage $s$ for any $k < e.$

Now, $a_{e+1,s}$ can only further change due to the action of $P_e.$ Arguing as we did in the preceding theorems, eventually $e$ such followers must get stuck in the gates $G_j$ for $j \leq e,$ at which point $P_e$ becomes met by the next follower. Now, $a_{e+1,s}$ cannot further change, (since changes only occur due to the action of $P_j$ for $j \leq e$ or by coding).

Hence we can now go to a stage $t_2 > t_1$ such that for all $j \leq e + 1$, for all $s > t_1,$

(i) $a_{j,s} = a_j,$
(ii) $P_j$ doesn't receive attention at stage $s.$

Let $j \leq e + 1.$ There are now two possibilities. Either $\forall s > t_1 \ (l(j, s) \leq a_{e+1}),$ in which case $r(e + 1, j, t_2) = r(e + 1, j),$ or $\exists s > t_1 \ (l(j, s) > a_{e+1}).$ In the latter case, at the least such stage $t_3,$ $r(e + 1, j, t_3) = r(e + 1, j),$ since at this stage all injurious numbers (which must have priority $< e + 1,$ are cleared). Hence $\lim_s r(e + 1, s) = r(e + 1)$ exists and is finite. By induction, we have (4.34). □

Finally, we need to show:

Lemma 4.35. If $W_e \sqcup V_e = A$ and $\Phi_e(B) = W_e,$ then $W_e$ is recursive.

Proof. We show how to compute if $x \in W_e$ or not. Let $s_0$ be a stage such that

(i) $l(e, s_0) > x,$
(ii) for all gates $G_j$ for $j \leq e$ if $G_j$ has a resident at stage $s_0$ then this resident is a permanent one,
(iii) $\forall j \leq e \forall s > s_0 \ (P_j$ doesn’t receive attention at stage $s),$
(iv) $\forall s > s_0 \ (f(s) > e).$

We claim $x \in W_e$ iff $x \in W_{e,s_0}.$ It is easy to see that at stage $s_0,$ if $x \notin W_{e,s_0}$ then as $l(e, s_0) > x, x = b_{j,s}$ for some $j > e.$ Also, by restraints since some number $< u(\Phi_{e,s_0}(B;x))$ must enter $B,$ this number must follow $P_k$ for some $k$ with $e \leq k < j.$

Choosing the first such number $y,$ this $y$ again must stop at gate $G_e$ or otherwise have been cancelled. But at the stage $y$ enters gate $G_e, b_{j,s}$ is enumerated into $A$ and $y$ remains at $G_e$ until $l(e, s) > ml(e, s) \geq l(e, s_0) > x.$ This must mean (by induction) $x$ must enter $V_e$ (since $\Phi_{e,x}(B;x)$ is holding $W_{e,x}(x)$) if $W_e \sqcup V_e$ really split $A.$ But then $x \notin W_e.$ □

We remark that Downey [8] has shown that not every r.e. degree contains an r.e. set with the antisplitting property. The classification of such degrees would appear difficult.
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References


**Note added in proof**

The first author has solved question (4.6) negatively (this is included in [8]). He has also solved question (3.8) affirmatively, and has classified the sublattices dense in $W$ as precisely the countable distributive ones. The results will appear in [37].