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Automorphisms of the Lattice of Recursively Enumerable Sets: Orbits*

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INTRODUCTION

An important program in the study of the structure of $\mathcal{E}$, the lattice of r.e. sets, is the classification of the orbits under $\text{Aut}(\mathcal{E})$, the automorphism group of $\mathcal{E}$. Obviously, the class of finite sets of a given cardinality and the class of infinite-coinfinitely recursive sets form orbits. But other than these, relatively few orbits are known. Harrington has shown that the creative sets form an orbit. The most important result in the area is still Soare's proof [So1] that maximal sets (sets which are coatoms in $\mathcal{E}^* = \mathcal{E}$ modulo the ideal of finite sets) form an orbit. The proof of this result contains an intricate and powerful construction which has since been used to construct automorphisms by Maass [M], Maass and Stob [MSt], and Stob [St]. It remains the most powerful method of constructing automorphisms.

The principal result of this paper is the existence of a new orbit, the hemimaximal sets. A r.e. set $H$ is \textit{hemimaximal} if there are a maximal set $M$ and disjoint r.e. sets $M_0, M_1$ such that $M_0 \cup M_1 = M$, each $M_i$ is non-recursive, and $M_0 = H$. In Section 1, we prove that the hemimaximal sets form an orbit by extending Soare's proof for maximal sets and detail the properties of splittings of maximal sets which led us to this result. In Section 2, we study the degrees of hemimaximal sets. (All degrees mentioned in this paper are r.e.) We show that hemimaximal sets exist in every high degree, and that for every degree $\alpha > 0$ there is a hemimaximal set $H$ such

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that \( \deg(H) \leq a \). However, there are some r.e. degrees which contain no hemimaximal sets. Thus the class of hemimaximal sets, \( \mathcal{H} \), provides an example of a definable (in \( \mathcal{E} \) in the language of lattices) class of r.e. sets which forms an orbit but contains r.e. sets of low r.e. degree. This is the first example of such an orbit. Furthermore, the class of degrees of sets in \( \mathcal{H} \) is not closed upward; again this is the first example of a class definable over \( \mathcal{E} \) with this property.

In Section 3, we turn to the question of which orbits contain complete r.e. sets. Soare has conjectured that the orbit of every nonrecursive r.e. set contains a complete set. Exploiting further our idea of finding automorphism types from splittings, we introduce the halfhemimaximal sets and show that every halfhemimaximal set is automorphic to a complete set. An r.e. set \( A \) is halfhemimaximal if \( A \) has a splitting into disjoint, nonrecursive, r.e. sets \( A_0, A_1 \) such that \( A_0 \) is hemimaximal. We show that, although not every set is halfhemimaximal, there are large natural classes of halfhemimaximal r.e. sets (such as all low simple r.e. sets).

In Section 4 we discuss the prospects of extending the idea of Section 1 to splittings of other sets which form orbits. Our main positive result is the classification of all orbits which arise from splitting quasimaximal sets (sets which are the intersection of finitely many maximal sets). We conclude in Section 5 with further remarks and a list of open questions.

Our notation is standard; a reference is Soare [So2]. In particular, a * affixed to a binary relation means up to a finite set; e.g., \( A \equiv^* B \) means \( A - B \) is finite. Also, we will use heavily the notion of e-states. If \( \{ V_\varepsilon \}_{\varepsilon \in \omega} \) is a recursive array of r.e. sets with simultaneous enumeration given by \( \{ V_{\varepsilon,s} \}_{\varepsilon,s \in \omega} \) then \( \sigma_{\varepsilon,s}(x) \), the e-state of \( x \) at \( s \), is \( \{ i \leq \varepsilon \mid x \in V_{i,s} \} \). The e-state of \( x \), \( \sigma_i(x) \), is \( \{ i \leq \varepsilon \mid x \in V_i \} \). e-states are ordered by the relation \( \sigma > \tau \) if and only if \( (\exists \varepsilon)(\forall i < \varepsilon)[(\varepsilon \in \sigma \Leftrightarrow \tau \in \varepsilon) \text{ and } i \in \sigma - \tau] \). Thus the empty set is the least e-state and \( \{ 0, 1, 2, ..., \varepsilon \} \) is the greatest. If we have a simultaneous enumeration of a recursive array of finite sets including \( A \) and \( B \), \( A \setminus B = \{ x \mid (\exists s)[x \in A_s - B_s] \} \) and \( A \searrow B \) denotes \( (A \setminus B) \cap B \).

1. HEMIMAXIMAL SETS FORM AN ORBIT

Definition 1. If \( A \) is any nonrecursive r.e. set, a nontrivial splitting of \( A \) is a pair of disjoint nonrecursive r.e. sets \( A_0, A_1 \) such that \( A = A_0 \cup A_1 \).

Friedberg was the first to show that every nonrecursive r.e. set \( A \) has a nontrivial splitting. In fact, the splittings that Friedberg produced had the slightly stronger property of Definition 2.

Definition 2. A nontrivial splitting \( A_0, A_1 \) of \( A \) is a Friedberg splitting
if for every r.e. \( W \), if \( W - A \) is not r.e., then each of \( W - A_0 \), \( W - A_1 \) is not r.e.

Sacks showed that nontrivial splittings \( A_0, A_1 \) of \( A \) always exist with the added property that \( A_0 <_T A \) and \( A_1 <_T A \) (and in fact it can be guaranteed that each of \( \text{deg}(A_0) \) and \( \text{deg}(A_1) \) is low).

**Definition 3.** A set \( A \) is hemimaximal if there are a maximal set \( M \) and a nontrivial splitting \( M_0, M_1 \) of \( M \) such that \( A = M_0 \). (More generally, if \( P \) is any property of r.e. sets, \( A \) is hemi\( P \) is there is a set \( M \) with property \( P \) and a nontrivial splitting \( M_0, M_1 \) of \( M \) such that \( A = M_0 \).)

Our investigation of hemimaximal sets was initially motivated by the following two theorems.

**Theorem 1.** Every nontrivial splitting of a maximal set is a Friedberg splitting.

*Proof.* Suppose that \( M_0, M_1 \) form a nontrivial splitting of the maximal set \( M \) and suppose that \( W - M \) is not r.e. Then \( W - M \) is infinite, so \( \overline{M} \subseteq \overline{W} \). Now \( W - M_0 \supseteq W - M = \overline{\overline{M}} \) so that if \( W - M_0 \) is r.e., \( \overline{M}_0 = \overline{M}_1 \cup (W - M_0) \) and so \( \overline{M}_0 \) is r.e. This contradicts the nonrecursiveness of \( M_0 \). Similarly, \( W - M_1 \) is not r.e. \( \square \)

**Definition 4 (Downey, [D]).** A pair of disjoint r.e. sets \( A_0, A_1 \) forms a maximal pair if whenever \( B_0, B_1 \) are such that \( B_i \supseteq A_i \), \( i = 0, 1 \), and \( B_0 \cap B_1 = \emptyset \) then \( B_i - A_i \) is finite for \( i = 0, 1 \).

**Theorem 2.** Every Friedberg splitting of a simple set forms a maximal pair.

*Proof.* Suppose that \( A \) is simple and that \( A_0, A_1 \) is a Friedberg splitting of \( A \), \( B_0 \supseteq A_0 \), \( B_1 \supseteq A_1 \), and \( B_0 \cap B_1 = \emptyset \). Now \( B_0 - A_1 = B_0 \) is r.e., thus \( B_0 - A \) is r.e. (since \( A_0, A_1 \) form a Friedberg splitting of \( A \)). Since \( A \) is simple, this implies that \( B_0 - A \) is finite. Similarly, \( B_1 - A \) is finite. \( \square \)

Theorems 1 and 2 led us to conjecture the following.

**Theorem 3.** If \( C, D \) are hemimaximal sets, then there is \( \Phi \in \text{Aut}(\mathcal{S}) \) such that \( \Phi(C) = D \).

*Proof.* Let \( A \) and \( B \) maximal be given such that \( A_0, A_1 \) and \( B_0, B_1 \) form nontrivial splittings of \( A \) and \( B \) respectively. Soare [So1] shows us how to construct \( \Phi \in \text{Aut}(\mathcal{S}) \) such that \( \Phi(A) = B \). We will modify his proof so that in addition \( \Phi(A_i) = B_i \) for each \( i \).
Soare constructs four recursive arrays, \( \{ U_e \}_{e \in \omega}, \{ V_e \}_{e \in \omega}, \{ \hat{U}_e \}_{e \in \omega}, \{ \hat{V}_e \}_{e \in \omega} \), such that the map \( \Phi \) defined by \( \Phi(U_e) = \hat{U}_e \) and \( \Phi^{-1}(V_e) = \hat{V}_e \) induces an automorphism of \( \mathcal{E}^* \). (It is enough to construct an automorphism of \( \mathcal{E}^* \) since Soare has also shown that if \( A \) and \( B \) are r.e. sets which are infinite and cofinite, and there is \( \Phi \in \text{Aut}(\mathcal{E}^*) \) such that \( \Phi \) maps the equivalence class of \( A \) to \( B \), then there is an automorphism of \( \mathcal{E} \) which maps \( A \) to \( B \).) To insure that \( \Phi \) defined in this way is defined on all of \( \mathcal{E}^* \) and is onto \( \mathcal{E}^* \), Soare guarantees that

\[
(\forall e)(\exists n)[ W_e = * U_n ] \quad \text{and} \quad (\forall e)(\exists n)[ W_e = * V_n ]. \tag{1}
\]

To guarantee that \( \Phi \) preserves inclusions, the only other requirement on \( \Phi \), Soare divides the problem into two subproblems, the so-called \( \hat{A} \) to \( \hat{B} \) part and the \( A \) to \( B \) part. To state exactly what each part requires, we need the following definition.

**Definition 5.** Let \( \{ X_e \}_{e \in \omega} \) and \( \{ Y_e \}_{e \in \omega} \) be recursive arrays of r.e. sets. The *full e-state*, \( v \) of \( x \) with respect to \( \{ X_e \}_{e \in \omega}, \{ Y_e \}_{e \in \omega} \) is the triple \( \langle e, \sigma, \tau \rangle \), where \( \sigma \) is the e-state of \( x \) with respect to \( \{ X_e \}_{e \in \omega} \) and \( \tau \) is the e-state of \( x \) with respect to \( \{ Y_e \}_{e \in \omega} \). (Given \( x \) and \( s, v_{e,s}(x) \) is the approximation to the full e-state of \( x \) at stage \( s \) in some fixed simultaneous enumeration of all the sets in the arrays \( \{ X_e \}_{e \in \omega}, \{ Y_e \}_{e \in \omega} \).)

Now the \( \hat{A} \) to \( \hat{B} \) part of the requirement amounts to

for each full e-state \( v \),

\[
\text{infinitely many elements of } \hat{A} \text{ have e-state } v \text{ with respect to } \{ U_e \}_{e \in \omega}, \{ \hat{U}_e \}_{e \in \omega}
\]

iff

\[
\text{infinitely many elements of } \hat{B} \text{ have e-state } v \text{ with respect to } \{ \hat{U}_e \}_{e \in \omega}, \{ V_e \}_{e \in \omega}.
\]

Similarly, the \( A \) to \( B \) requirement is

for each full e-state \( v \),

\[
\text{infinitely many elements of } A \text{ have e-state } v \text{ with respect to } \{ U_e \}_{e \in \omega}, \{ \hat{V}_e \}_{e \in \omega}
\]

iff

\[
\text{infinitely many elements of } B \text{ have e-state } v \text{ with respect to } \{ \hat{U}_e \}_{e \in \omega}, \{ V_e \}_{e \in \omega}.
\]
It is clear that (1), (2), and (3) guarantee that \( \Phi \) as defined above is an automorphism such that \( \Phi(A) = *B \). Now for our theorem, it is enough to replace (3) by

for each \( i \) and for each full e-state \( v \),

infinitely many elements of \( A_i \) have e-state \( v \) with respect to

\[ \{U_e\}_{e \in \omega}, \{\hat{U}_e\}_{e \in \omega} \]

iff

infinitely many elements of \( B_i \) have e-state \( v \) with respect to

\[ \{\hat{U}_e\}_{e \in \omega}, \{V_e\}_{e \in \omega} \].

The most difficult of the three conditions (1), (2), and (3) is (3) (and our (4)). The primary reason for this difficulty is the conflict between (2) and (3). To see why this is so, suppose that \( U_0 \) is given. (Suppose for instance, because of (1), that \( U_0 \) is enumerated to satisfy \( U_0 = \mathcal{W}_0 \).) Then, as we observe elements in \( U_0 \cap A \), we must enumerate certain elements in \( \hat{U}_0 \) while they remain in \( B \). However, these elements may later enter \( B \) thereby threatening (3) with respect to \( U_0 \). For if \( \hat{U}_0 \cap B \) is infinite, we must have that \( U_0 \cap A \) is infinite but we have no control over \( U_0 \). Thus a necessary condition for meeting (3) seems to be that if infinitely many elements enter \( B \) while in \( \hat{U}_0 \), infinitely many elements of \( A \) must be in \( U_0 \). Soare extends this analysis to all e-states to get a sufficient condition on the enumeration on all the sets in the four arrays above for (3) to be met. A preliminary definition is needed.

**Definition 6.** Given full e-states \( v = \langle e, \sigma, \tau \rangle \) and \( v' = \langle e, \sigma', \tau' \rangle \), \( v \leq v' \) if \( \sigma \leq \sigma' \) and \( \tau \supseteq \tau' \). (The relation \( \leq \) is pronounced "is covered by.")

**Lemma 1 (Soare's Extension Lemma).** Assume that \( A \) and \( B \) are infinite r.e. sets and \( \{U_n\}_{n \in \omega}, \{V_n\}_{n \in \omega}, \{\hat{U}_n\}_{n \in \omega}, \{\hat{V}_n\}_{n \in \omega} \) are recursive arrays of r.e. sets. Suppose that there is a simultaneous enumeration of a recursive array including all the above such that \( A \setminus \hat{V}_n = \emptyset = B \setminus \hat{U}_n \) for all \( n \). For each full e-state \( v \) with respect to \( \{U_n\}_{n \in \omega}, \{\hat{V}_n\}_{n \in \omega} \), define the r.e. set \( D^A_v = \{x \mid x \in A_{s+1} - A_s \) for some \( s \) such that \( v_{e,s}(x) = v \} \). Similarly, define \( D^B_v \) with \( B, \{\hat{U}_n\}_{n \in \omega}, \{\hat{V}_n\}_{n \in \omega} \) in place of \( A, \{U_n\}_{n \in \omega}, \{V_n\}_{n \in \omega} \). Furthermore suppose that

\[ (\forall v)[D^A_v \text{ infinite } \Rightarrow (\exists v')[(v \leq v' \text{ and } D^A_{v'} \text{ infinite})]] \]

and

\[ (\forall v)[D^A_v \text{ infinite } \Rightarrow (\exists v')[(v' \leq v \text{ and } D^B_{v'} \text{ infinite})]]. \]
Then there are r.e. sets $\tilde{U}_n$ extending $\tilde{U}_n$ and $\tilde{V}_n$ extending $\tilde{V}_n$ such that (3) above is satisfied.

The first step then in our proof is to extend the Extension Lemma to pairs of r.e. sets to guarantee that (4) above is met. To this end, define $D^A_n$ as $D^A_n$ above but with $A_i$ in place of $A$ for each $i = 0, 1$. Similarly, define $D^B_n$. Then we have

**Lemma 2.** Let $A$ and $B$ be infinite r.e. sets and $A_0$, $A_1$ and $B_0$, $B_1$ form splittings of $A$ and $B$, respectively. Suppose that $\{U_n\}_{n \in \omega}$, $\{V_n\}_{n \in \omega}$, $\{\tilde{U}_n\}_{n \in \omega}$, $\{\tilde{V}_n\}_{n \in \omega}$ are recursive arrays of r.e. sets and that there is a simultaneous enumeration of a recursive array including all the above such that $A_i \psi \tilde{V}_n = \emptyset = B_i \psi \tilde{U}_n$, for all $n$ and $i$. Furthermore suppose that for each $i$, $i = 0, 1$, 

$$(\forall v)[D^R_v \text{ infinite} \Rightarrow (\exists v') [v \leq v' \text{ and } D^A_v \text{ infinite}]]$$

and

$$(\forall v)[D^A_v \text{ infinite} \Rightarrow (\exists v') [v' \leq v \text{ and } D^R_v \text{ infinite}]].$$

Then there are r.e. sets $\tilde{U}_n$ extending $\tilde{U}_n$ and $\tilde{V}_n$ extending $\tilde{V}_n$ such that (4) above is satisfied.

**Proof of Lemma 2.** Apply Soare's Extension Lemma (Lemma 1) to the pair $A_0$, $B_0$, in place of $A$, $B$. The extension guaranteed there meets (4) with respect to $A_0$ and $B_0$. Further, the proof of Soare's Extension Lemma guarantees that $\tilde{U}_n - \tilde{U}_n \subseteq B_0$ and $\tilde{V}_n - \tilde{V}_n \subseteq A_0$ for all $n$. Now, renaming the sets $\tilde{U}_n$ and $\tilde{V}_n$ which result from this application of Lemma 1 to $\tilde{U}_n$ and $\tilde{V}_n$, we see that the hypotheses of the Lemma 1 are now satisfied with $A_1$ and $B_1$ in place of $A$ and $B$. Thus, applying the Extension Lemma again, we get that (4) is satisfied with respect to $A_1$, $B_1$.

It remains to meet (1) and (2) and satisfy the hypotheses of Lemma 2. Obviously, (1) can be satisfied by taking $\tilde{U}_e = W_e$ and $\tilde{V}_e = W_e$ for all $e$. However, to facilitate meeting (2) and the hypotheses of Lemma 1, Soare chooses the sets $U_n$ and $V_n$ to have certain further properties. He proves

**Lemma 3.** Let $M$ be maximal. Then there is a simultaneous enumeration of a recursive array $\{Z_n\}_{n \in \omega}$ satisfying (1), with $Z_0 = M$, and such that

$$(\forall n)[Z_n \setminus M \text{ infinite} \iff Z_n - M \text{ infinite} \iff Z_n \supset M]$$

and

$$(\forall n)(\forall m) > n (a.a. x)[x \in (Z_n \setminus M) \cap (Z_m \setminus M) \Rightarrow x \in (Z_n \setminus Z_m)]$$

where $a.a. x$ denotes for almost all $x$. 
We shall choose \( \{ U_n \}_{n \in \omega} \) and \( \{ V_n \}_{n \in \omega} \) as the arrays resulting from the application of Lemma 3 to \( A \) and \( B \) respectively. Thus, (1) is satisfied. It remains to meet (2) and the hypotheses of the Extension Lemma. Now for maximal sets, (2) amounts to

\[ (\forall n)[\bar{A} \subseteq U_n \iff \bar{B} \subseteq * \bar{U}_n] \]

and similarly with \( B \) and \( V_n \) in place of \( A \) and \( U_n \). Thus, to complete the proof, we need to prove the following.

**Lemma 4.** Given maximal sets \( A \) and \( B \), let \( \{ U_n \}_{n \in \omega} \) and \( \{ V_n \}_{n \in \omega} \) be the recursive arrays given by Lemma 3 for \( A \) and \( B \), respectively. Then there exist recursive arrays \( \{ \bar{U}_n \}_{n \in \omega}, \{ \bar{V}_n \}_{n \in \omega} \) together with a simultaneous enumeration of a recursive array including all of \( \{ U_n \}_{n \in \omega}, \{ V_n \}_{n \in \omega}, \{ \bar{U}_n \}_{n \in \omega}, \) and \( \{ \bar{V}_n \}_{n \in \omega} \) which satisfy the hypotheses of Lemma 2 and also (2) which here asserts that

\[ [\bar{A} \subseteq U_n \iff \bar{B} \subseteq * \bar{U}_n] \quad \text{and} \quad [\bar{B} \subseteq V_n \iff \bar{A} \subseteq * \bar{V}_n]. \]  

**Proof.** This proof follows the proof of the analogous lemma of Soare [So1, Theorem 3.2] very closely. At the end of stage \( s \) we will have defined \( A_s \) and \( B_s \). Given \( A_s \) and \( B_s \), define \( \psi_s(x) \) to be the function which maps \( A_s \) onto \( B_s \) in increasing order. Let \( \psi = \lim_s \psi_s \). We will use \( \psi \) to govern our enumeration of the sets \( \bar{U}_n \) and \( \bar{V}_n \). We must enumerate \( \bar{U}_n \) and \( \bar{V}_n \) to satisfy (9). We will thus attempt to guarantee that \( x \in U_n \iff \psi(x) \in \bar{U}_n \) and similarly for \( \bar{V}_n \). The two problems with this strategy are that we only have an approximation \( \psi_s(x) \) to \( \psi(x) \) at any stage \( s \), and that we must not enumerate so much in \( \bar{U}_n \) so as to violate the hypotheses of the Extension Lemma. Actually, the first problem is minor; the second is handled by not enumerating an element \( y \) in \( \bar{U}_n \) until the state to which it will be raised is successfully covered by elements entering both \( A_0 \) and \( A_1 \).

**Construction.** Stage \( s = 4t \). Let \( x \) be the unique element enumerated in some \( U_e \) at stage \( t \) in the given simultaneous enumeration of \( \{ U_n \}_{n \in \omega} \). Enumerate \( x \) in \( U_{e,s} \). Let \( A_s = U_{0,s} \).

Stage \( s = 4t + 1 \). As in stage \( 4t \) with \( V_n \) for \( U_n \) and \( B \) for \( A \).

Stage \( s = 4t + 2 \). Let \( (x', e') \) be the first pair \( (x, e) \) with the following properties P1–P4. If there exist such a pair, enumerate \( \psi_{s-1}(x') \in \bar{U}_{e',s} \). Otherwise do nothing at this stage.
\[(x \in U_{e,s-1} - A_{s-1}) \quad \text{and} \quad (\psi_{s-1}(x) \notin \bar{U}_{e,s-1}) \quad (P1)\]
\[(\forall i)_<e \ [x \in U_{i,s-1} \iff \psi_{s-1}(x) \in \bar{U}_{i,s-1}] \quad (P2)\]
\[(\forall i)_<e \ [x \in \bar{V}_{i,s-1} \iff \psi_{s-1}(x) \in V_{i,s-1}]. \quad (P3)\]

Define
\[
\sigma_0 = \{i \mid i \leq e \text{ and } \psi_{s-1}(x) \in \bar{U}_{i,s-1}\}
\]
\[
\tau_0 = \{i \mid i \leq e \text{ and } \psi_{s-1}(x) \in V_{i,s-1}\}
\]
\[
u_x = (\text{the unique } u)_<e \ [x \in U_{e,u+1} - U_{e,u}]
\]
\[(\forall i)_{i=0,1} (\exists u)(\exists v)(\exists \sigma_1)(\exists \tau_1)[u_x < v < s-1 \text{ and } y \in A_{i,v+1} - A_{i,v} \text{ and } \sigma_1 \supseteq \{e\} \cup \sigma_0 \text{ and } \tau_1 \subseteq \tau_0 \text{ and } \sigma_1 = \{i \mid i \leq e \text{ and } y \in U_{i,v}\} \text{ and } \tau_1 = \{i \mid i \leq x \text{ and } y \in \bar{V}_{i,v}\}]. \quad (P4)\]

Stage \(s = 4t + 3\). Similarly, attempt to enumerate some element \(\psi_{s-1}(x') \in \bar{V}_{e,s}\), where \(B_i, U_n, \bar{U}_n, V_n, \bar{V}_n\) and \(\psi_{s-1}(x)\) in the preceding stage are replaced by \(B_i, V_n, \bar{V}_n, U_n, \bar{U}_n\) and \(\psi_{s-1}(x)\). Also, in P3, replace \((\forall i)_{i<e}\) by \((\forall i)_{i\leq e}\) to reflect the priority of \(\bar{U}_e\) over \(\bar{V}_e\).

**Claim 1.** The hypotheses of the Extension Lemma, Lemma 2, are met by the above arrays and simultaneous enumeration.

**Proof of Claim 1.** It is obvious that \(B_i \supseteq \bar{U}_n = \emptyset = A_i \supseteq \bar{V}_n\) for all \(n\) and \(i\) since we enumerate only elements \(\psi_{s-1}(x)\) in \(\bar{U}_n\). By “speeding up” the enumeration of \(A\) if necessary, we may assume that infinitely many \(x\) appear in each \(A_i\) before appearing in any \(U_j, j > 0\), or \(\bar{V}_j\). Thus we need only verify (5) and (6) for states \(\sigma \neq \emptyset\). We verify (5); the verification of (6) is dual.

Fix \(i, e\), and the full e-state \(v_0 = \langle e, \sigma_0, \tau_0 \rangle\) with \(\sigma_0 \neq \emptyset\). Assume that \(D^{A_i}_{v_0}\) is finite for all \(v_1 \geq v_0\). We must show that for each \(\tau_1 \subseteq \tau_0\), only finitely many \(x\) whose e-state with respect to \(\{V_n\}_{n \in \sigma_0}\) is \(\tau_1\) are allowed to enter \(\bar{U}_{\sigma_0} = \bigcap \{\bar{U}_n \mid n \in \sigma_0\}\). Since e-states increase with time it follows that \(D^{B_i}_{v_0}\) must be finite also.

Let \(e_0 = \max \sigma_0\). It follows by P2 of the construction and (8) of Lemma 3 that \(\bar{U}_{e_0} \supseteq \bar{U}_n\) is finite for all \(n < e_0\). Hence, if there are infinitely many \(x\) in \(\bar{U}_{e_0}\), almost all such \(x\) enter \(\bar{U}_{e_0}\) only after \(x \in \bar{U}_n\), for all \(n \in \sigma_0 - \{e_0\}\). But by P4 of the construction and our assumption that \(D^{A_i}_{v_1}\) is finite for all \(v_1 \geq v_0\), at most finitely many \(x\) already in \(\bigcap \{\bar{U}_n \mid n \in \sigma_0 - \{e_0\}\}\) will be allowed to enter \(\bar{U}_{e_0}\) while the e-state of \(x\) with respect to \(\{V_n\}_{n \in \sigma_0}\) is some \(\tau_1 \subseteq \tau_0\).

**Claim 2.** (9) is satisfied.
Proof of Claim 2. It is only here that we use that \( A_0, A_1 \) and \( B_0, B_1 \) are nontrivial splittings. This implies that if \( U \) is any r.e. set such that \( U - A \) is infinite, then \( U \searrow A_i \) is infinite for each \( i \). (Otherwise, since \( U - A = * A_i \searrow \cup U \) and hence \( A_i \) is r.e.)

We prove (9) by induction on \( e \). Assume then that (9) is true for \( i < e \).
If \( U_e \cong \bar{A} \) then \( U_e \setminus A \) is finite by (7) and hence \( \bar{U}_e \setminus B \) and \( \bar{U}_e - B \) are finite. Assume then that \( U_e \cong \bar{A} \).
Choose \( y_0 \) such that for all \( x \in \bar{A} \), \( x \geq y_0 \) implies

\[
(\forall i < e)[[x \in U_i \iff \psi(x) \in \bar{U}_e] \text{ and } [x \in \bar{V}_i \iff \psi(x) \in V_e]].
\]

We claim that for any \( x_0 \in \bar{A} \), \( x_0 \geq y_0 \) implies \( \psi(x_0) \in \bar{U}_e \). Define \( \sigma_0 = \{i \mid i < e \text{ and } U_i \equiv \bar{A}\} \) and define

\[
U_{\sigma_0} = \bigcap \{U_i \mid i \in \sigma_0\}.
\]

Now \( U_{\sigma_0} \equiv \bar{A} \) and hence \( U_{\sigma_0} \searrow A_i \) must be infinite for \( i = 0, 1 \) by the property of nontrivial splittings of \( A \) mentioned at the beginning of the proof of this claim. Define

\[
\tau_0 = \{i \mid i \leq x_0 \text{ and } \psi(x_0) \in V_i\}.
\]

Now if \( i \leq x_0 \) and \( i \notin \tau_0 \), then \( V_i - B \) is finite and hence \( V_i \setminus B \) is finite by (7). But for any \( i, V_i \setminus B \) finite implies \( \bar{V}_i \setminus A \) finite. Hence there exist \( \sigma_1 \equiv \sigma_0 \) and \( \tau_1 \subseteq \tau_0 \), such that

\[
(\forall i)_{i=0,1} (\forall t)(\exists v)(\exists y)[ y \in A_{i,v+1} - A_{i,v} \text{ and } \sigma_1 = \{i \mid i \leq e \}
\text{ and } y \in U_{i,v}] \text{ and } \tau_1 = \{i \mid i \leq x_0 \text{ and } y \in \bar{V}_{i,v}\}].
\]

But then by P4, \( \psi(x_0) \) is eventually enumerated in \( \bar{U}_e \). The case of \( \psi^{-1}(V_e) = * \bar{V}_e \) is handled similarly.

The combination of Lemmas (2) and (3) establishes Theorem 1.

2. Degrees of Hemimaximal Sets

The degrees of maximal sets are precisely the high degrees. One goal of this work was to find an orbit in which a much larger class of degrees is represented. It is not known, for instance, whether there is an orbit in which every nonzero r.e. degree is represented.

Definition 7. \( H = \{\text{deg}(A) \mid A \text{ is hemimaximal}\} \).

We summarize the results of this section:
(a) For every nonzero \( c \) r.e., there is \( a < c \) such that \( a \in H \) (Theorem 4).

(b) \( H_1 \leq H \) (Theorem 5).

(c) There is a nonzero degree \( c \), such that \( c \notin H \) (Theorem 6).

(d) For all \( a < b \), if \( b \in \mathbb{L}_1 \), there is a degree \( c \notin H \) such that \( a \leq c \leq b \) (Theorem 7).

By (a) and (b), \( H \) is an orbit class which contains more than only high degrees. (An orbit class is a class of r.e. degrees determined by an orbit of \( \text{Aut}(\mathcal{S}) \).) This is the first example of such an orbit class. All previously known orbits consisted of only recursive sets or only high sets. By (a) and (d), \( H \) is not closed upwards in the r.e. degrees; this is the first example of any definable (in \( \mathcal{S} \)) class of r.e. sets with this property. We turn now to the proofs of the theorems mentioned above.

**Theorem 4.** Suppose that \( \emptyset <_T C \). Then there is a hemimaximal set \( A \), such that \( \emptyset <_T A \leq_T C \).

**Proof.** We will construct \( M \) maximal and \( A, B \) a nontrivial splitting of \( M \) such that \( A \) is nonrecursive and \( A \leq_T C \). \( A <_T C \) is guaranteed by the usual technique of permitting. \( M \) will be constructed by the usual e-state construction of a maximal set. The requirements, including those to make \( A \) nonrecursive, are thus as follows:

- \( N_e : |\tilde{M}| \geq e \);
- \( P_e : W \neq \tilde{A} \);
- \( Q_e : \tilde{M} \) has almost constant e-state.

The priority ranking is \( Q_0 < N_0 < P_0 < Q_1 \cdots \). At the end of stage \( s \), the elements of \( \tilde{M} \) are denoted by \( m_0 \prec m_1 \prec \cdots \). Note that the requirements \( P_e \) guarantee that \( A \) is nonrecursive. To ensure as well that \( B \) is nonrecursive we may assume that \( C \) has low r.e. degree. This is enough since \( \deg(A) \cup \deg(B) = \deg(M) \) and \( \deg(M) \) is high.

**Construction.** **Stage \( s + 1 \).** Step 1. \( (P_e) \) Let \( c \) be the least element enumerated in \( C \) at stage \( s + 1 \). Let \( e \) be the least integer (if any) such that \( W_e \cap A = \emptyset \) and there is an \( x \) satisfying

- (a) \( x \in W_{e,s} \cap \tilde{M}_s \),
- (b) \( x \notin \{ m_0 \prec \cdots \prec m_e \} \) (priority),
- (c) \( x > c \) (permitting).

Enumerate the least such \( x \) in \( A \).
Step 2. (Q_1) Find the least i (if any) and the least j for i such that 
\sigma_{i,s}(m_i^j) < \sigma_{i,s}(m_j^i) and if x was enumerated in A at step 1, 
m_j^i < x. Enumerate m_i^j in B.

**Lemma 0.** \( A \preceq_T C \).

**Proof.** If s is a stage such that \( C_s[x] = C[x] \), then \( A_s[x] = A[x] \).

**Lemma 1.** For every e, \( N_e \) is satisfied.

**Proof.** \( N_e \) is injured by each of \( P_0, P_1, \ldots, P_{e-1} \) at most once each. It is 
injured thereafter by \( Q_i, i \leq e \) at most \( 2^i \) times after it has ceased being 
injured by \( Q_j, j < i \). Thus \( N_e \) is injured only finitely often and is satisfied.

**Lemma 2.** For every e, \( Q_e \) is satisfied.

**Proof.** We show that for all \( j > i \geq e \), if \( m_i = \lim s m_i^j \) and \( m_j = \lim s m_j^i \) 
then \( \sigma_e(m_i) \geq \sigma_e(m_j) \). (This implies that \( Q_e \) is satisfied; since there are only 
finitely many e-states the sequence \( \sigma_e(m_i) \geq \sigma_e(m_{e+1}) \geq \sigma_e(m_{e+2}) \geq \ldots \) 
must have a limit.) If this is not true, there must be cofinitely many s such 
that \( \sigma_{e,s}(m_i^j) < \sigma_{e,s}(m_j^i) \), \( m_i^j = m_i \) and \( m_j^i = m_j \). But at any such stage, \( m_i^j \) 
would be enumerated in B at step 2.

**Lemma 3.** For every e, \( P_e \) is satisfied.

**Proof.** Otherwise, let \( e \) be the least counterexample; we show that \( C \) is 
recursive, contrary to hypothesis. Let \( \sigma \) be the e-state guaranteed by 
Lemma 2; \( e \in \sigma \) since \( W_e = \bar{A} \). Let \( i_0 \) be such that \( j \geq i_0 \) implies that \( m_j = \lim s m_j^i \) 
has e-state \( \sigma \). Let \( s_0 \) be such that \( j \leq i_0 \Rightarrow m_j^{s_0} = m_j \) and \( \sigma_{e,s}(m_j^{s_0}) = \sigma \).

**Claim.** Suppose that \( s > s_0 \) is such that \( x = m_i^j, j \geq i_0, \) and \( \sigma_{e,s}(x) = \sigma \). 
Then for all \( t \geq s \) there is \( y \geq x \) such that \( y = m_j^i, j \geq i_0, \) and \( \sigma_{e,s}(y) = \sigma \).

(The claim implies the lemma, since to answer \( k \in C \), find \( s \geq s_0 \) such that 
there is \( x \geq k \) such that x satisfies the hypotheses on x in the claim. Then 
by the claim, \( C \) does not permit below \( x \) after stage \( s \) or else \( x \) or some 
element \( y \geq x \) is available at that stage to be enumerated in A at step 1.)

**Proof of the Claim.** If \( x \notin \bar{M} \), then \( x \) satisfies the conclusion for all \( t \). 
Otherwise, let \( t \) be such that \( x \) enters \( M \) at stage \( t \) (necessarily in step 2). 
Then \( x = m_i^j \) and there is \( y = m_j^i, j > i \) such that \( \sigma_{i,s}(x) < \sigma_{i,s}(y) \). Then \( y > x \) 
since \( j > i \) and \( \sigma_{e,s}(y) = \sigma_{e,s}(x) = \sigma \) else \( m_i^j \) would be enumerated in B 
instead of x. Thus \( y \) satisfies the conclusion at \( t \). Now if \( y \in \bar{M} \), then \( y \) 
satisfies the conclusion of the claim for all later \( t \) else we can repeat the 
above argument with \( y \) in place of \( x \).
THEOREM 5. For every high r.e. degree \( a \), there is a hemimaximal set \( A \) such that \( \text{deg}(A) = a \).

Proof. Our proof is a slight modification of Lachlan's version of Martin's proof (see [So2, Theorem XI.2.3]) that every high degree contains a maximal set. Given a high r.e. degree \( a \), there is a r.e. set \( D \) of degree \( a \), and an enumeration \( \{D_s\}_{s \in \omega} \) of \( D \) such that if \( \bar{D} = \{d_0 < d_1 < d_2 \ldots \} \), \( \bar{D}_s = \{d_s^0 < d_s^1 < \ldots \} \), and \( c_D(x) = (\mu s)[d_x^s = d_x] \), then \( c_D \) dominates every total recursive function. We will construct disjoint r.e. sets \( A \) and \( B \) such that \( M = A \cup B \) is maximal, \( M \equiv_T D \), \( B \) is nonrecursive, and \( B \leq_T A \). Thus \( A \) will be our desired hemimaximal set of degree \( a \). The requirements are the usual maximal set requirements.

\( Q_e \): \( \bar{M} \) has almost constant \( e \)-state, and the requirements to make \( B \) nonrecursive,

\( P_e \): \( B \neq W_e \).

In addition we must ensure that \( B \leq_T A \) and \( A \equiv_T D \). As in Theorem 4, \( \bar{M}_s = \{m_s^0 < m_s^1 < \ldots \} \).

CONSTRUCTION. Stage \( s + 1 \). Step 1 (To satisfy \( Q_e \).) Given \( e \leq i < j \), we say \( m_i^e \) is attracted to \( m_j^e \) for \( Q_e \) if

\[
\sigma_{e-1,s}(m_i^e) = \sigma_{e-1,s}(m_j^e),
\]

(10)

\[
\sigma_{e,s}(m_i^e) < \sigma_{e,s}(m_j^e),
\]

(11)

and

\[
(\forall e' < e) [W_{e'} \supseteq \{m_i^e, \ldots, m_j^e\} \Rightarrow W_{e',s} \cap B_s \neq \emptyset].
\]

(12)

(The condition in (12) reflects the priority of \( P_e \) over \( Q_e \) for \( e' < e \).) Choose the least \( i \) such that there are \( e, i, \) and \( j \) such that \( m_i^e \) is attracted to \( m_j^e \) for \( Q_e \) and such that

\[
(\exists y \leq m_i^e)[d_y^{s+1} \neq d_y^s].
\]

Let \( e \) be least for \( i \) and \( j \) least for \( e \). Enumerate each of \( m_i^e, \ldots, m_{j-1}^e \) into \( A_{s+1} \). (Thus \( m_i^{s+1} = m_{i+k}^e \) for all \( k \geq 0 \).)

Step 2. Let \( z = (\mu y)[d_y^{s+1} \neq d_y^s] \). If some \( x \leq m_{2z+3}^e \) was enumerated in \( A \) at step 1, do nothing more. Else enumerate \( m_{2z+2}^e \) in \( A_{s+1} \) and \( m_{2z+3}^e \) in \( B_{s+1} \) unless one of these two integers has a higher \( 2z \)-state than both of \( m_{2z}^e \) and \( m_{2z+1}^e \). In that case, enumerate \( m_{2z}^e \) in \( A_{s+1} \) and \( m_{2z+1}^e \) in \( B_{s+1} \). (The purpose of the choice in step 2 is this: if some \( m_i^e \) for
\(i < 2z\) is attracted to at least one of \(m_{2z}^i, m_{2z+1}^i, m_{2z+2}^i,\) or \(m_{2z+3}^i\), then it will be attracted to at least one of the two of these elements which remain in \(\mathcal{M}\) after step 2).

**Lemma 1.** For each \(e\), \(\lim_s m^i_s = m_e\) is finite.

**Proof.** By induction on \(e\). Suppose that for \(i < e\) there is a stage \(s_0\) such that \(m_i = m^i_{s_0}\). Let \(t > s_0\) be such that \(d^i_t = d_e\). Then step 2 never causes \(m^i_e\) to change at any stage \(s > t\). However, step 1 causes \(m^i_e\) to change after \(s_0\) only to increase its \(e\)-state. This happens \(2^e\) times at most.

**Lemma 2.** \(D \equiv^T M, B \leq^T A\)

**Proof.** \(D \leq^T M\) by step 2; if \(m^i_{2e+3} = m^i_{2e+3}\), then \(d^i_e = d_e\). \(M \leq^T D\) since if \(d^i_e = d_e\), then \(e \in M\) iff \(e \in M_s\).

\(B \leq^T A\) since elements enter \(B\) only at step 2 and when such an integer enters \(B\), a smaller integer enters \(A\) at the same stage.

**Lemma 3.** For each \(e\),

1. \(Q_e\) is met,
2. \(W_e \supseteq \mathcal{M} \Rightarrow W_e \cap B \neq \emptyset\). (This implies \(P_e\).)

**Proof.** The proof is by induction on \(e\). Suppose then that the lemma is true for \(i < e\). This implies, by (a), that there are an \((e - 1)\)-state \(\sigma_0\) and an \(i_0\) such that for all \(i \geq i_0\), \(\sigma_{e-1}(m_i) = \sigma_0\). Let \(s_0\) be such that if \(i \leq i_0\), then \(m_{i_0} = m_i\). By (b), we may also assume that for all \(i < e\), either \(W_{i, i_0} \cap B_{i_0} \neq \emptyset\) or \((\exists j \leq i_0)(m_j \notin W_i)\). The significance of this last assumption is that clause (12) of the definition of attraction for \(Q_e\) will hold for all \(m^i_{s}\) where \(i \geq i_0\) and \(s \geq s_0\).

Now assume for a contradiction that \(W_e \cap \mathcal{M}\) and \(\overline{W}_e \cap \mathcal{M}\) are both infinite; let \(\sigma_1 = \sigma_0 \cup \{e\}\).

Define a recursive function \(h\) as follows. If \(x \leq m_{i_0}\), let \(h(x) = 0\). For \(x > m_{i_0}\), define

\[
h(x) = (\mu y)(x \in M_s \text{ or } \sigma_{e,s}(x) > \sigma_0 \text{ or } \exists i)(x = m^i_{s} \text{ and } \sigma_{e,s}(m^i_{s}) = \sigma_0 \text{ and } (\exists j > i)(\sigma_{e,s}(m^j_{s}) > \sigma_1)]\]

(If the last clause of (13) holds, then \(x = m^i_{s}\) is attracted to \(m^j_{s}\) for \(Q_e\) for some \(e' \leq e\).)

**Claim.** If \(h(x)\) is defined by the last clause of (13), \(x \in \mathcal{M}\), and \(\sigma_e(x) = \sigma_0\), then \(d^h(x) = d_x\).
To see that the claim gives the desired contradiction, note that the conclusion implies that \( c_P(x) < h(x) + 1 \). However, there are infinitely many \( x \) satisfying the hypotheses of the claim; namely, any \( x \in W_e \cap M \) which is not one of \( m_0, m_1, \ldots, m_{k_0} \). Thus \( c_P(x) \) does not dominate the recursive function \( h(x) + 1 \).

**Proof of Claim.** Fix \( x \) and assume that the hypothesis of the claim holds for \( x \). We will show that

\[
(\forall s \geq h(x)) (\exists k) [x < m^*_k \text{ and } \sigma_1 \leq \sigma_{e,s}(m^*_k)].
\]

Now (14) implies the conclusion of the claim since it implies that for all \( s \), there is an \( e' < e \) such that \( x \) is attracted to some \( y > x \) for the sake of \( Q_{e'} \) at all stages \( s \geq h(x) \). We prove (14) by induction on \( s \geq h(x) \). For \( s = h(x) \), (14) is satisfied with \( k = f \). Suppose \( s \geq h(x) \) and \( m^*_k \) satisfies (14) for \( s \) but not for \( s + 1 \). There are two cases.

**Case 1.** \( m^*_k \in M_{s+1} \) by step 1. Then there are \( p \) and \( q \), \( p < q \), such that step 1 applies with \( i = p \) and \( j = q \). Now \( p \leq k \) and \( k \neq q \) since \( m^*_k \in M_{s+1} \). Also \( x < m^*_p \) since \( x \notin M_{s+1} \). Thus \( \sigma_{e,s}(m^*_p) \geq \sigma_{e,s}(m^*_k) \) since otherwise we would have chosen \( j = k \) in step 1. Thus (14) holds at \( s + 1 \) via \( m^*_a = m^*_{p+1} \).

**Case 2.** \( m^*_k \in M_{s+1} \) by step 2. Then (14) still holds at \( s + 1 \) by virtue of the remark made at the end of step 2 of the construction.

We turn now to the proof of (b). Suppose that \( W_e \supseteq M \) and \( B \cap W_e = \emptyset \). Then, using (a) for \( i \leq e \), there are \( i_1 \) and \( s_1 \) such that

\[
(\forall i \geq i_1)[\sigma_{e}(m_i) = \sigma_1],
\]

\[
(\forall i \leq i_1)[m^*_i = m_i],
\]

\[
(\forall i \leq i_1)[\sigma_{e,s_1}(m_i) = \sigma_e(m_i)].
\]

Similar to the proof of Theorem 4, we have the following claim.

**Claim.** Suppose that \( j \) and \( s_2 \) are such that \( 2j + 3 \geq i_1, s_2 \geq s_1 \), and for all \( i \), if \( i_1 \leq i \leq 2j + 3 \), then \( \sigma_{e,s_2}(m^*_i) = \sigma_1 \), then, for all \( s \geq s_2 \), \( m^*_i = m^*_{2j+3} = m^*_{2j+3} \).

**Proof of Claim.** Suppose to the contrary that \( s \) is least and \( k \) least for \( s \) such that \( i_1 \leq k \leq 2j + 3 \) and \( m^*_k \neq m^*_{k+1} \). There are two cases.

**Case 1.** Step 1 applied at \( s + 1 \) with \( i = k \). Now \( m^*_k \) can only be attracted to \( m^*_j \) for some \( Q_{e'}, e' \leq e \) for the conclusion of (12) does not hold with \( e \) in place of \( e' \). Thus, we have that \( \sigma_{e,s+1}(m^*_{k+1}) > \sigma_1 \). But then we
can argue by induction on \( t > s \) that for all \( t \), there is \( j \) such that \( i_1 < j \leq k \) for which \( \sigma_{e,t}(m_j^t) > \sigma_1 \). This contradicts the choice of \( i_1 \).

**Case 2.** Step 2 applied at \( s + 1 \). In this case, \( m_k^t \) is enumerated in \( A_{s+1} \) and \( m_{k+1}^t \) is enumerated in \( B_{s+1} \). Since \( k \) is even, \( k + 1 \leq 2j + 3 \). Thus, \( m_{k+1}^t \in W_{e,s} \). This gives a contradiction since \( W_e \cap B \) was assumed to be empty.

The claim implies part (b) of the lemma for since there are infinitely many \( j \) satisfying the hypotheses of the claim, the conclusion implies that \( M \) is recursive, contradicting Lemma 2.

We remark that not just any maximal set \( M \) will have the property of \( M \) in Theorem 5, namely that there is a nontrivial splitting \( A, B \) of \( M \) such that \( A \equiv_T M \). The methods of [DW] may be used to construct a maximal set \( M \) such that if \( A, B \) is a nontrivial splitting of \( M \), then \( A \) and \( B \) form a minimal pair.

**Theorem 6.** There is an r.e. set \( C \) such that if \( A \equiv_T C \), then \( A \) is not hemimaximal.

**Proof.** Let \((\Phi_e, \Gamma_e, U_e, V_e)_{e \in \omega}\) be an effective listing of all quadruples where \( \Phi_e, \Gamma_e \) are recursive functionals and \( U_e, V_e \) are disjoint r.e. sets.

Then the requirements on \( C \) amount to the following:

\[
R_e: \quad \Phi_e(C) = U_e \text{ and } \Gamma_e(U_e) = C \text{ implies } U_e \cup V_e \text{ is not maximal.}
\]

We will attempt to ensure that \( U_e \cup V_e \) is not maximal by enumerating an array \( \{T_{e,i}\}_{i \in \omega} \) of disjoint, finite, r.e. sets such that \( T_{e,i} \not\subseteq U_e \cup V_e \). This guarantees that not only is \( U_e \cup V_e \) not maximal, it is not even hyperhyper-simple. Thus the requirements \( R_e \) will be divided into the following:

\[
R_{e,i}: \quad \Phi_e(C) = U_e \text{ and } \Gamma_e(U_e) = C \text{ implies } T_{e,i} \cap U_e \cup V_e \neq \emptyset.
\]

We will assume that the requirements are ordered in some \( \omega \)-sequence, thereby inducing a priority ordering on them. We first give the strategy for meeting a single requirement; it will be convenient in describing it to drop all subscripts. The requirement thus becomes:

\[
R: \quad \Phi(C) = U \text{ and } \Gamma(U) = C \text{ implies } T \cap U \cup V \neq \emptyset.
\]

Let \( \phi(x, s) \) and \( \gamma(x, s) \) be the use functions associated with the computations \( \Phi_s(C_s; x) \) and \( \Gamma_s(U_s; x) \), respectively, and let \( l_{\phi}(s) \) and \( l_{\gamma}(s) \) be the corresponding lengths of agreements of these functions.

To attack \( R \) we proceed as follows.
Step 1. Wait for a stage $s$ such that there is an $x$ with

1. $l_R(s) > x$,
2. $(\exists y \leq \gamma(x, s))[y \notin U_s \cup V_s \cup T_s]$, and
3. $l_{\psi}(s) > \gamma(x, s)$.

(Such $x$ and $s$ must exist if the hypotheses of $R$ are satisfied and $U \cup V$ is not cofinite.) Given $x$ and $s$, our action is to enumerate into $T_{s+1}$ all $y \leq \gamma(x, s)$ such that $y \notin U_s \cup V_s$ and to restrain from $C$ all $z \leq \phi(\gamma(x, s), s)$.

Note that after step 1, $R$ is satisfied temporarily since $T_{s+1} \cap U_s \cup V_s \neq \emptyset$. $R$ will be satisfied forever (with the finite restraint imposed by step 1) unless there is a stage $t > s$ such that $(U_t \cup V_t) \supseteq T_t = T_{s+1}$. Now if any element, say $z$, of $T_{s+1} - (U_s \cup V_s)$ is enumerated into $U_t - U_s$, we have, by the restraints imposed on $C$ at step 1, that

$$\Phi_t(C_t; z) = \Phi_s(C_s; z) = U_s(z) \neq U_t(z)$$

and this disagreement is preserved forever with finite restraint. Thus we may assume that each element enumerated in $T$ at stage $s+1$ is later enumerated into $V$ by stage $t$. Now at stage $t+1$ we perform

Step 2. Remove the restraint on $C$ imposed by step 1. Enumerate $x$ (the $x$ of step 1) into $C_{t+1}$.

Step 2 wins requirement $R$ forever since we have that

$$\Gamma(U; x) = \Gamma_s(U_s; x) = C_s(x) \neq C_{t+1}(x).$$

The first equality is the crucial one and is true since $U_s[\gamma(x, s)] = U[\gamma(x, s)]$ because $U$ and $V$ are disjoint sets, $U_t \cup V_t \supseteq \{x \mid x \leq \gamma(x, s)\}$, and $U_t = U_s$.

To see that the strategies for the various $R_{e, i}$ cohere, note that each $R_{e, i}$ imposes only finite restraint on $C$ and thus $R_{e, i}$ may be restarted for the sake of $R_{e', i'}$ of higher priority as in standard arguments of Friedberg–Muchnik type. The only restraints on the sets $T_{e, i}$ are to make $T_{e, i}$ disjoint from $T_{e, j}$ if $i \neq j$ and it is clear that this can be done. We will omit the details of combining the strategies for meeting the $R_{e, i}$ since this is a straightforward application of the finite injury priority method.

The strategy for meeting requirement $R$ of the previous theorem can be combined with other techniques to provide various strengthenings of the result. For example, we can also insure that $c$ contains no hemi-$r$-maximal set. Another example is
THEOREM 7. \( H \) is dense in the low r.e. degrees (i.e., if \( a, b \in L_1 \), then there is \( c \) such that \( a \leq c \leq b \) and \( c \) contains no hemimaximal set).

Proof. We have added two requirements on \( C \) to those of the previous theorem. First, we require that \( \text{deg}(C) \leq b \). This is easy to arrange via the standard technique of permitting. Note that step 2 of the construction of Theorem 6, the only step requiring enumeration in \( C \), does not have to be performed at stage \( t+1 \). In fact, at any stage \( v \geq t+1 \), the situation still exists for diagonalization and if \( x \) is enumerated in \( C \) at stage \( v \), \( R \) is satisfied. This is precisely the characteristic of a construction necessary for combination with permitting.

The second additional requirement on \( C \) is that \( \text{deg}(C) \geq a \). Let \( A \) be any fixed set of degree \( a \). Requirement \( R \) now reads

\[
R: \quad \Phi(C \oplus A) = U \text{ and } I(U) = C \implies T \cap U \cup V \neq \emptyset.
\]

Given that \( a \) is low, there is a standard technique (due to Robinson [R]) for meeting a requirement of the form \( R \). For a complete description and examples of the use of this technique we refer the reader to Soare [So2, Chap. 12, Theorem 3.1]. Here we give an informal description. The principal difficulty in meeting this new version of requirement \( R \) is that we now longer have complete control in step 1 to restrain the computation \( \Phi_s(C \oplus A_s) = U_s \) since we have no power to restrain \( A \). Robinson showed that if \( C \) is low, there is a way to “certify” computations as being “\( A \)-correct” in such a way that incorrect computations are certified only finitely often. Specifically, if \( A \) is low,

\[
\{ j \mid (\exists n)[n \in W_j \text{ and } D_n \cap A = \emptyset] \} \leq_T \emptyset'.
\]

Here \( D_n \) is the finite set with canonical index \( n \). (15) guarantees the existence of a recursive function \( g \) such that

\[
\lim_{\delta} g(j, s) = \begin{cases} 1 & \text{if } (\exists n)[n \in W_j \text{ and } D_n \cap A = \emptyset] \\ 0 & \text{otherwise.} \end{cases}
\]

We use \( g \) as follows. Suppose we wish to perform step 1 in the above construction. Then we enumerate \( n \) in an otherwise empty r.e. set \( W \), where \( D_n \) is the set of numbers used negatively by \( A \) in the computation. By the recursion theorem, we assume that we know the index \( j \) for \( W \). We then search for a stage \( t > s \) such that \( D_n \cap A \neq \emptyset \) or \( g(j, t) = 1 \). In the former case, we do not believe the computation and do not start an attack. In the latter case, we say that the computation is “certified,” and start the attack as above. Now it may be the case that the computation was not \( A \)-correct; we then repeat the process using the same set \( W \). The point is that by (15),
we will not certify infinitely many incorrect computations so that finally we
will be justified in believing a computation and then the argument for R
proceeds just as before. A more detailed description of a use of lowness is
given in the next section in Theorem 12.

There are no additional conflicts imposed by combining this lowness
strategy and permitting. As these techniques, separately and in combination,
are now quite standard and there are no surprises in their combination to
meet all the requirements R, we omit further details.

By employing techniques of Ambos-Spies and Fejer [ASF, Theorem 3.3], we can extend Theorem 7 to the following.

**Theorem 8.** There is an interval \([a, b]\) of low r.e. degrees containing no
hemimaximal r.e. set. In fact, every interval of low r.e. degrees contains an
interval with no hemimaximal set.

3. **Orbits Containing Complete Sets**

The question of just what sets are automorphic to complete sets is still
open. Soare has conjectured that every r.e. set can be taken to a complete
set by an automorphism. (Since creativeness is invariant under
automorphisms, not every set is automorphic to an incomplete set.)
Harrington (unpublished) has shown that not every r.e. set can be taken to
a complete set by an automorphism \(\Phi\) which is \(\Sigma_3\)-presented (i.e., the fun-
c tion \(f\) such that \(\Phi(W_e) = W_{f(e)}\) is \(\Sigma_3\)). The significance of this is that all
automorphisms constructed using Soare's machinery, such as that of
Section 1 of the current paper, are \(\Sigma_3\)-presented. Since there are complete
hemimaximal sets, every hemimaximal set is automorphic to a complete
set. Using the hemimaximal sets as a base and exploiting the idea of classi-
fying r.e. sets by the types of nontrivial splittings that they have, we were
able to extend this result to a much larger class of r.e. sets.

**Definition 8.** An r.e. set \(A\) is **halfhemimaximal** if there is a nontrivial
splitting \(A_0, A_1\) of \(A\) such that \(A_0\) is hemimaximal.

More generally, if \(P\) is any property of r.e. sets, \(A\) is **half\(P** if there is a
nontrivial splitting \(A_0, A_1\) of \(A\) such that \(A_0\) has property \(P\).

The importance of halfhemimaximality for this program is the following
theorem.

**Theorem 9.** If \(A\) is halfhemimaximal, then there is a \(B \equiv_T \emptyset'\) and
\(\Phi \in \text{Aut}(\mathcal{S})\) such that \(\Phi(A) = B\).
Let $A_0, A_1$ be a nontrivial splitting of $A$ with $A_0$ hemimaximal. Let $B_0$ be a complete hemimaximal set (which exists by Theorem 5). Let $\Phi \in \text{Aut}(\mathcal{A})$ be such that $\Phi(A_0) = B_0$. Now $B = \Phi(A)$ is our desired complete set since $\mathcal{O}' \equiv_T B_0 \leq_T B$. (The last inequality is since $B = B_0 \cup \Phi(A_1)$ and $B_0$ and $\Phi(A_1)$ are disjoint. 1

Though we will show below that many sets are halfhemimaximal, it is easy to see from the following theorem that there are sets which are not.

**Theorem 10.** If $A$ is halfhemimaximal, then $A$ has a maximal superset.

**Proof.** Let $A_0, A_1$ be a nontrivial splitting of $A$ and $B_0, B_1$ a nontrivial splitting of a maximal set $B$ with $B_0 = A_0$. We claim that $A \subseteq^* B$. Otherwise, since $B$ is maximal, $A \subseteq^* B$. Thus $B_0 =^* A_1 \cup B_1$ and so $B_0$ is recursive contradicting its being a part of a nontrivial splitting. 1

**Corollary 1.** Every non-low$_2$ degree contains a set which is not halfhemimaximal.

**Proof.** Shoenfield [Sh] has shown that every nonlow$_2$ degree contains a set with no maximal superset. The result then follows directly from the theorem. 1

We can improve Theorem 10 and its corollary with a little more work.

**Theorem 11.** Every nonzero r.e. degree contains a nonhalfhemimaximal set.

**Proof.** Suppose that we are given a nonrecursive set $B$. We will construct $A$ nonhalfhemimaximal such that $B \equiv_T A$. Let $g$ be a 1–1 recursive function such that $g(\omega) = B$. Let $\{F_x\}_{x \in \omega}$ be a recursive sequence of disjoint finite sets such that $\bigcup \{F_x\}_{x \in \omega} = \omega$ and $|F_x| = x + 2$ for every $x$. We ensure that $A \equiv_T B$ in the following way. At stage $s$ of the construction, we will enumerate exactly one element into $A$ chosen from $F_{g(s)}$. Thus $|A \cap F_x| \leq 1$ and $|A \cap F_x| = 1$ iff $x \in B$. It is easy to see from this that $A \equiv_T B$.

The requirements that $A$ be nonhalfhemimaximal can be put as follows. Let $(U_e, V_e)_{e \in \omega}$ be a recursive listing of all pairs of disjoint r.e. sets. Then the requirements are

$N_e$: $U_e \subseteq A$ and $U_e \cup V_e \not\subseteq A \Rightarrow U_e \cup V_e$ is not maximal or $U_e$ is recursive.

(The requirements are enough by Theorem 10. In fact they show something slightly stronger about $A$ than only nonhalfhemimaximality.) We may assume in the light of the hypotheses of $N_e$ that no element of $U_e$ is enumerated in $U_e$ before it is enumerated in $A$. 1
CONSTRUCTION. Stage s. Let \( x = g(s) \). We must enumerate one element of \( F_x \) into \( A \). Choose the least element \( z \) from \( F_x \) such that for all \( e \leq x \), \(|F_x \cap V_{e,s}| \geq x + 1\) implies that \( z \in V_{e,s} \). This is possible since \(|F_x| = x + 2\) and the condition on \( z \) requires it only to be in the intersection of at most \( x + 1 \) subsets of \( F_x \) each of cardinality at least \( x + 1 \). This intersection is nonempty. Enumerate \( z \) into \( A \).

To complete the verification we need only show that \( N_e \) is satisfied for each \( e \). So suppose that the hypotheses of \( N_e \) are satisfied and that \( U_e \subset V_e \) is maximal. We will show that \( U_e \) is recursive. Since \( U_e \subset V_e \) is maximal, there is an integer \( x_0 \) such that for all \( x \geq x_0 \), \(|F_x \cap (U_e \cup V_e)| \geq x + 1\) (see [Ro, Chap. 12, Theorem XIII]). Let \( z \) be fixed such that \( z \in F_x, x \geq x_0 \). We show how to decide if \( z \in U_e \).

Let \( s \) be a stage such that \(|F_x \cap V_{e,s}| \geq x + 1\) or \(|F_x \cap A_s| = 1\). One or the other must happen since if \( F_x \cap A = \emptyset \) then \(|F_x \cap V_e| \geq x + 1\). In the former case, \( z \notin U_e \) since if \( z \) is later enumerated in \( A \), \( z \in V_e \). In the latter case, if \( z \in A \) we can enumerate \( U_e \) and \( V_e \) until \( z \) appears in one or the other (this must happen since \( A \subseteq U_e \cup V_e \)) and if \( z \notin A \) then \( z \notin U_e \).

Despite Theorem 11, there are large natural classes of r.e. sets consisting entirely of halfhemimaximal sets.

**Theorem 12.** The following are halfhemimaximal:

(a) every low\(_2\) simple set,
(b) every semilow\(_{1.5}\) simple set,
(c) every d-simple set with a maximal superset.

**Proof.** Robinson [R] proved that every low r.e. set has a maximal superset. Lachlan [L] and Bennison and Soare [BSo] extended this result and the technique used in proving it to low\(_2\) and semilow\(_{1.5}\) sets, respectively. We will show how to extend Robinson's method to prove

(d) every low simple r.e. set is halfhemimaximal.

The same device we use to prove (d) can be used to extend the Lachlan and Bennison–Soare methods to prove (a) and (b).

Suppose then that \( A \) is low and simple. We will construct \( B \supseteq A \), \( B \) maximal, which witnesses that \( A \) is halfhemimaximal. We briefly review the Robinson construction of \( B \). Our exposition follows Soare [So2]. We have a set of movable markers \( A_e, e \in \omega \). The position of \( A_e \) at the end of stage \( s \), \( A_{e,s} \), is the \( e \)th element of \( B_s \). Thus, since we will always have \( A_s \subseteq B_s \), we make \( B \) coinfinite by meeting for every \( e \) the requirements

\[ R_e : \lim A_{e,s} < \infty. \]
The usual strategy for making $B$ maximal is to move $A_{e,s}$ to maximize its $e$-state. This guarantees that the following requirements are met:

\[ Q_e: \text{almost all elements of } B \text{ have the same } e\text{-state.} \]

In the usual construction of a maximal set (as in Theorem 4), at stage $s$ we move $A_e$ to a new element of the complement of $B$, only if that new element has higher $e$-state than $A_{e,s}$. This requires us to move $A_e$ only finitely often (at most after $A_0, A_1, \ldots, A_{e-1}$ have ceased moving). Further, it guarantees $Q_e$ since the $e$-states of the final positions of the markers $A_e, A_{e+1}, \ldots, A_{e+k}, \ldots$ are nonincreasing in $k$.

This strategy conflicts with the requirement $A \subseteq B$ in the following way. $A_0$, for instance is moving so as to maximize its $0$-state. Thus, if $(\exists x)[x \in B \text{ and } x \in W_{0,s}]$, $A_{0,s}$ will move to $x$ if $A_{0,s} \notin W_{0,s}$. However, later $x$ may be enumerated in $A$ causing $A_0$ to move again. We are thus tempted to move $A_0$ infinitely often if $W_0 \setminus A$ is infinite but $W_0 - A = \emptyset$.

To avoid this problem, we need to use the lowness of $A$ to give us advice on which elements to which $A_e$ might move are actually elements of $A$. The lowness of $A$ guarantees that there is a recursive function $f$ such that

\[
\lim_{s} f(e, s) = \begin{cases} 
1 & \text{if } W_e \cap A \neq \emptyset \\
0 & \text{if } W_e \cap A = \emptyset.
\end{cases}
\]

The oracle $f$ is used in the following way. Suppose that we are attempting to move $A_e$ to $x$ to raise its $e$-state to $r$. We then enumerate $x$ in a certain test set $U_\sigma$. By the recursion theorem, we may assume that we know an index for $U_\sigma$, say $U_\sigma = W_j$. Using $f$, we locate a stage $t \geq s$ such that

\[ U_\sigma \subseteq A_t \quad \text{or} \quad f(j, t) = 1. \]

By the properties of $f$, $t$ must exist. Now only in the latter case do we move $A_e$ to $x$. Of course it may still be that $x$ is later enumerated in $A$; we then perform the same procedure with another $x'$ which is in $e$-state $\sigma$. The point is that $A_e$ will not be moved infinitely often by this process in an attempt to find an element in $e$-state $\sigma$. For, if $A_e$ is moved infinitely often to an element in $e$-state $\sigma$, we must have that $\lim_s f(j, s) = 1$ and so $U_\sigma \cap A \neq \emptyset$. But any element of $U_\sigma \cap A$ will remain as $A_e$ forever. Of course the above argument presumes that no higher priority $A_j$ or $\sigma' > \sigma$ intervenes. In that case, we say that $U_\sigma$ is injured and we begin anew with a "fresh" (empty) version of $U_\sigma$.

To make $A$ halfhemimaximal, it suffices to enumerate $B_0, B_1$ such that $B_0 \subseteq A$, $B_0 \cup B_1 = B$, $B_0 \cap B_1 = \emptyset$, and $B_0$ nonrecursive. To see that this is enough, define $A_0 = B_0$ and $A_1 = B_1 \cap A$. Obviously $A_0, A_1$ split $A$ and $A_0$ is nonrecursive. $A_1$ is nonrecursive since otherwise $B - A = B_1 - A_1$ is r.e.
contradicting the simplicity of $A$. ($B - A$ is infinite since no low r.e. set is maximal.) Finally, $B_1$ is nonrecursive since $B_0 \leq_T A$ and $B \equiv_T B_0 \oplus B_1$. (The former implies that $B_0$ is low, and thus the latter implies that $B_1$ is nonrecursive since $B$ is high.)

We define $B_0$ and $B_1$ as follows

$$B_0 = \{x \mid \exists s(x \in A_{s+1} - B_s)\},$$

$$B_1 = \{x \mid \exists s(x \in B_{s+1} - A_{s+1})\}.$$ 

Informally, $B_0$ is the set of those $x$ which are enumerated in $A$ before we have decided to enumerate them in $B$. $B_0, B_1$ satisfy all of the above requirements except perhaps the requirement that $B_0$ is nonrecursive. Thus we have the following requirements on $B_0$:

$$S_e: B_0 \neq \bar{W}_e.$$ 

$S_e$ will function as a negative requirement on $B$ as follows. The ideal way for $S_e$ to be satisfied is for an element of $W_e \cap B$ to be enumerated in $A$ before we enumerate it in $B$. This element is then in $B_0 \cap W_e$ and $S_e$ is satisfied forever. Thus our strategy will be to restrain elements of $W_e$ from $B$. If we restrain an infinite r.e. set $W$ of such elements from $B$, we must have that $W \cap A \neq \emptyset$ by the simplicity of $A$ giving us the desired witness to $W_e \cap B_0 \neq \emptyset$. This restraint is accomplished in the following construction by giving requirement $S_e$ priority over all but $N_0, N_1, \ldots, N_e$. Thus, we are entitled to enumerate a marker position, $A_{i,s}$ for $i \geq e$ into $B$ only if by so doing it will increase its e-state (rather than only its i-state as is usual) if $W_{e,s} \cap B_{0,s} = \emptyset$ and $A_{i,s} \in W_{e,s}$. We now give the details of the construction.

We assume that $W_0 = A$. During the course of the construction, we will speed up the enumeration of $A$. Our enumeration at the end of stage $s$ will be denoted $A_s$. We will always have $W_{0,s} \subseteq A_s$. Our construction will have for each stage $s + 1$, substages $e$ for $0 \leq e \leq s + 1$. During stage $s + 1$, $A_{s+1}$ will denote the result in our enumeration of $A$ at the beginning of the current substage. Given $U$ an r.e. set, to test $U$ at stage $s$ is to do the following. Compute an index $e$ of $U$ from the construction. (This uses the Recursion Theorem.) Find $t \geq s$ such that

$$U_s \subseteq W_{0,t} \cup A_s \quad \text{or} \quad f(e, t) = 1.$$ 

Such a $t$ always exist since $\lim_s f(e, s) = 1$ if $W_e \cap \bar{A} \neq \emptyset$. In the latter case $f(e, t) = 1$, we say that the test succeeds. For all sets $U$ which we test in the construction, we will have exactly one element $x$ in $U_s - A_s$. If the test fails (necessarily because $x \in W_{0,s}$), we will enumerate $x$ in $A_s$. For each $e$ and e-state $\sigma$, we will define an r.e. set $U_{e,\sigma}$. These r.e. sets will serve as test sets
for the marker \( A_e \). These sets \( U_{e,\sigma} \) will be enumerated uniformly in \( e, \sigma \) and so we will be entitled to test them in the sense described above. Furthermore, from time to time we shall reset \( U_{e,\sigma} \); to reset \( U_{e,\sigma} \) at stage \( s \) means to replace \( U_{e,\sigma} \) by a "fresh" empty version of \( U_{e,\sigma} \). When \( U_{e,\sigma} \) is referred to in the construction below, the current version is meant.

**Construction.** **Stage 0.** Define \( A_{e,s} = e \) for all \( e \).

**Stage \( s + 1 \).** **Substage \( e, 0 \leq e \leq s \).** If \( e = 0 \) enumerate all of \( W_{0,s+1} \) into \( A_{s+1} \). Substage \( e \) occurs only if \( A_{0,s}, \ldots, A_{e-1,s} \) have not moved during this stage. In this case, we determine whether \( A_e \) wants to move during this stage (to some \( x \) for the sake of some \( \sigma \)). There are two cases. Let \( z = A_{e,s} \).

**Case 1.** \( z \in A_{s+1} \). Then \( A_e \) wants to move at \( s + 1 \). \( \alpha \) and \( x \) are determined as follows. In order of decreasing \( \sigma \), \( e \)-state, we search for a \( \sigma \) for which the following procedure results in a successful test of \( U_{e,\sigma} \).

Given \( \sigma \), enumerate in the current version of \( U_{e,\sigma} \) the least \( x > z \) such that \( x \notin B_s \cup A_{s+1} \) and \( \sigma_{e,s}(x) = \sigma \). If no such \( x \) exists, proceed to the next \( \sigma \) else test \( U_{e,\sigma} \). If the test succeeds, we have found \( \sigma \) and \( x \); otherwise we choose the next least \( x \) satisfying \( x > z \) and \( \sigma_{e,s}(x) = \sigma \), enumerate it in \( U_{e,\sigma} \), and test \( U_{e,\sigma} \). If \( \sigma \) is not the empty \( e \)-state, this procedure must eventually end with a successful test or no further \( x \) in \( e \)-state \( \sigma \). If \( \sigma \) is the empty \( e \)-state this procedure must end with a successful test since otherwise a cofinite set is enumerated for \( U_{e,\sigma} \), and so \( U_{e,\sigma} \cap \overline{A} \neq \emptyset \) but \( \lim_s f(i,s) = 0 \), where \( i \) is the index of \( U_{e,\sigma} \). This contradicts the properties of \( f \).

**Case 2.** \( z \notin A_s \). Let \( \sigma' = \sigma_{e,s}(z) \). For each \( \sigma > \sigma' \), we do the following. We first determine whether \( z \) is restrained by some \( S_i \) for \( \sigma \). Let \( j \) be the least integer such that \( j \in \sigma - \sigma' \). Then \( z \) is restrained for \( \sigma \) by \( S_i \) if \( i < j \), \( i \in \sigma' \) and \( W_{i,s} \cap A_s = \emptyset \). If \( z \) is not restrained for \( \sigma \) we perform the procedure described in case 1 for determining whether \( \sigma \) and \( x \) satisfy the condition of \( A_e \) wanting to move to \( x \) for the sake of \( \sigma \). This test will be finite since for each such \( \sigma \) there are only finitely many \( x \) in state \( \sigma \) at stage \( s + 1 \). This case may terminate without \( A_e \) wanting to move to any \( x \).

If \( A_{e,s} \) wants to move to \( x \) for either of the above two reasons, we move \( A_e \) by enumerating into \( B_{s+1} \) all \( y \) such that \( A_{e,s} \leq y < x \). (Thus \( x = A_{e,s+1} \).) Further, we reset all \( U_{e',\sigma} \) such that \( e'>e \) or \( e'=e \) and \( \sigma' < \sigma \).

**Substage \( s + 1 \).** At substage \( s + 1 \), we enumerate all of \( A_{s+1} \) into \( B_{s+1} \).

**Lemma 1.** \( \lim_s A_{e,s} \) exists, and so \( \overline{B} \) is infinite.

**Proof.** The proof is by induction on \( e \). Let \( s_0 \) be the least stage such that \( A_{i,s_0} = \lim_s A_{i,s} \) for all \( i < e \). At \( s_0 \), \( U_{e,\sigma} \) is reset for every \( \sigma \). Assume for a contradiction that \( A_e \) moves infinitely often; each time \( A_e \) moves after \( s_0 \),
it moves because it wants to. Let \( \sigma \) be the maximal e-state such that \( A_e \) wants to move for the sake of \( \sigma \) infinitely often. Let \( s_1 \geq s_0 \) be the least stage such that after \( s_1 \), \( A_e \) never wants to move for any \( \sigma' > \sigma \). Then \( U_{e,\sigma} \) is reset at \( s_1 \). Now for each stage \( s \geq s_1 \) at which \( A_e \) moves for the sake of \( \sigma \) to \( x \), there is \( t > s \) such that \( f(i, t) = 1 \), where \( i \) is the index of the current (final) version of \( U_{e,\sigma} \). Thus, \( \lim_{s} f(i, s) = 1 \) so that \( U_{e,\sigma} \cap A \neq \emptyset \). However by the construction, there is at any stage \( s \) at most one element of \( U_{e,\sigma} - A \), and that element becomes the marker position at the end of stage \( s \). Thus, there is \( s \geq s_1 \) such that \( A_{e,s} \in U_{e,\sigma} - A \) and that element remains the marker position of \( A_e \) forever. This contradiction establishes the lemma.

**Lemma 2.** For every \( e \), \( W_e \neq B_0 \), only finitely many elements of \( B \) are permanently restrained from \( B \) by \( S_e \), and there is an \((e + 1)\)-state \( \sigma \) such that almost every element \( x \in B \) has \((e + 1)\)-state \( \sigma \).

**Proof.** The proof is by induction on \( e \). Suppose then that the lemma is true for \( i < e \). Then there is an \( e \)-state \( \sigma \) such that almost all elements of \( B \) are in \( e \)-state \( \sigma \). Therefore, either \( e \notin \sigma \) or \( e \in \sigma \). In the former case, obviously \( W_e \neq B \) and only finitely many elements of \( B \) are permanently restrained from \( B \) by \( S_e \). (Any element restrained by \( S_e \) is an element of \( W_e \)). Suppose then that \( e \in \sigma \). Let \( i_0 \) and \( s_0 \) be such that

\[
(\forall i > i_0)[A_i \text{ has final e-state } \sigma],
\]

\[
(\forall i \leq i_0)[A_{i,s_0} = \lim_{s} A_{i,s}],
\]

\[
(\forall i \leq i_0)[\sigma_{i,s_0}(A_{i,s_0}) = \sigma_i(A_{i,s_0})].
\]

Define an r.e. set \( W \) as follows:

\[
W = \{ x \mid (\exists i \geq i_0)(\exists s \geq s_0)\} x = A_{i,x}, \text{ and } (\forall j)[i_0 < j < i \Rightarrow \sigma_{x,j}(A_{i,x}) = \sigma \}.
\]

It is obvious that \( W \) is r.e. and that \( B \subseteq^* W \). Suppose that \( W_e = B_0 \). This implies that no element of \( W \) enters \( A \) before it enters \( B \). We claim however that no element of \( W \) is enumerated in \( B \) before it is enumerated in \( A \). This would provide the desired contradiction since \( W \) is infinite and \( A \) is simple. Suppose then to the contrary that \( x \) is the least element of \( W \) that enters \( B \), say at stage \( s \). Then \( x = A_{i,x} \) for some \( i > i_0 \), \( \sigma_{e,s}(x) \geq \sigma \), and \( x \) enters \( B \) because it wants to (else there is some smaller element of \( W \) entering \( B \) at stage \( s \)). Since \( x \) wants to enter \( B \), necessarily by Case 2, there is \( y = A_{e,y} \) such that \( j > i \) and \( \sigma_{j,x}(A_{i,x}) < \sigma_{j,y}(A_{e,y}) \). But \( x \) is restrained from \( B \) at stage \( s \) by \( S_e \) so in fact \( \sigma_{e,s}(x) < \sigma_{e,s}(y) \). But then \( A_{i_0} \) would want to move to \( y \) at substage \( i_0 \) of stage \( s \) (or else \( y \) would be enumerated in \( A \)) since \( \sigma_{e,s}(A_{i_0,e}) = \sigma < \sigma_{e,s}(y) \). Thus \( x \) never wants to move. This contradiction establishes that
We prove that if $B \subseteq W_e$, then $W_e \cap B \neq \emptyset$. Thus in this case too only finitely many elements are restrained from $B$ by $S_e$. To see that the final clause of Lemma 2 is correct, let $\sigma_0$ and $\sigma_1$ be the two $(e+1)$-states which can appear infinitely often in $\overline{B}$; $\sigma_0 = \sigma$ and $\sigma_1 = \sigma \cup \{ e+1 \}$. Suppose that $x = \lim A_{i,s}$, $y = \lim A_{j,s}$, where $i < j$ and $\sigma_0 = \sigma_{e+1}(x) < \sigma_{e+1}(y) = \sigma_1$ and $x$ and $y$ are not permanently restrained from $B$ by $S_0, \ldots, S_e$. Then $A_i$ would want to move to $y$ at infinitely many stages and so would move. Thus we cannot have infinitely many such pairs $x$ and $y$ and so there is an $(e+1)$-state (either $\sigma_0$ or $\sigma_1$) in which almost every element of $\overline{B}$ resides.

Turning now to the proof of part (c) of the theorem, recall the definition of $d$-simple.

**Definition 9.** An r.e. set $A$ is $d$-simple if for every r.e. set $X$ there is an r.e. set $Y \subseteq X$ such that $X \cap \overline{A} = Y \cap \overline{A}$ and if $Z$ is any r.e. set with $Z - X$ infinite, then $(Z - Y) \cap A \neq \emptyset$.

So suppose $A$ is $d$-simple and has a maximal superset $M$. Let $X$ in the above definition be $M$. Given the resulting $Y$, let $M_1 = Y \setminus A$ and $M_0 = A - (Y \setminus A)$. Since $Y \cap \overline{A} = M \cap \overline{A}$, $M_0, M_1$ form a splitting of $M$. To see that $M_0$ is nonrecursive, suppose that $Z = M_0$. Then $Z - M$ is infinite so $(Z - Y) \cap A \neq \emptyset$. But $(Z - Y) \cap A \subseteq M_0$. This of course is a contradiction.

To complete the proof, there are two cases.

Case 1. $M_1$ is nonrecursive. Then $M_0, M_1$ form a nontrivial splitting of $M$ and thus $M_0 \cap A$ and $M_1 \cap A$ form a nontrivial splitting of $A$ with $M_0 \cap A = M_0$ hemimaximal witnessing that $A$ is halfhemimaximal.

Case 2. $M_1$ is recursive. Then let $B_0, B_1$ be a nontrivial splitting of $M_0$. Let $M_0' = B_0$ and $M_1' = M_1 \cup B_1$. $M_0', M_1'$ form a nontrivial splitting of $M$ and as in Case 1, $M_0' \cap A$ and $M_1' \cap A$ witness that $A$ is halfhemimaximal.

4. **Orbits of hemiP Sets**

Various strong (but false) conjectures might be made after examining the result of section 1 that hemimaximal sets form an orbit. We had hoped that the following might be true.

**Conjecture 1 (False).** If $P$ is a property of r.e. sets such that $\mathcal{P} = \{ A \mid A$ has $P \}$ forms an orbit, then the hemiP sets form an orbit.

That conjecture 1 is false is easy to see. Perhaps the most striking counterexample is the property of creativeness. The counterexample relies on the following very easy theorem.
THEOREM 13. If \( K \) is a creative set and \( C \) is any r.e. set, then \( K \oplus C \) (\( = \{2x \mid x \in K\} \cup \{2x + 1 \mid x \in C\} \)) is creative.

Proof. We have \( K \leq_1 K \oplus C \) by \( f(x) = 2x \). Thus \( K \oplus C \) is 1-complete and hence creative.

Now the creative sets form an orbit. However if \( C \) is any nonrecursive r.e. set, \( A_0 = \{2x \mid x \in K\} \) and \( A_1 = \{2x + 1 \mid x \in C\} \) form a nontrivial splitting of \( K \oplus C \). It is obvious that not all sets \( A_i \) which arise in this way are automorphic. For instance, \( A_1 \) may be maximal in an infinite, coinfinite recursive set (if \( C \) is maximal) or nowhere simple (if, say, \( C \) is nowhere simple).

One promising possibility motivated by Theorem 2 is to weaken Conjecture 1 by requiring that the splittings be Friedberg splittings and that the property \( P \) imply simplicity. Thus, we weaken Conjecture 1 to

Conjecture 2. Let \( \mathcal{C} \) be a class of r.e. simple sets which forms an orbit. Then

\[ \{A \mid A \text{ is half of a Friedberg splitting of a set in } \mathcal{C}\} \]

forms an orbit.

We do not have much evidence in either direction for Conjecture 2. Obviously the problem is that we do not know many orbits. However, we are able to establish Conjecture 2 for the next easiest case to the hemimaximal sets.

DEFINITION 10. An r.e. set \( Q \) is \( k \)-quasimaximal if \( Q \) is the intersection of exactly \( k \) maximal sets which are pairwise infinitely different.

Soare showed, by applying his result on maximal sets, that the \( k \)-quasimaximal sets form an orbit. We establish Conjecture 2 for the quasimaximal sets by proving the following.

THEOREM 14. Let \( A \) and \( B \) be \( k \)-quasimaximal and \( A_0, A_1 \) and \( B_0, B_1 \) be Friedberg splittings of \( A \) and \( B \), respectively. Then there is \( \Phi \in \text{Aut}(\mathcal{C}) \) such that \( \Phi(A_0) = B_0 \).

Proof. We give the proof only for \( k = 2 \). The case \( k > 2 \) is similar. Let \( M_0, M_1 \) be maximal sets such that \( M_0 \cap M_1 = A \); we may assume that \( M_0 \cup M_1 = \omega \). Similarly, let \( N_0, N_1 \) be maximal sets such that \( N_0 \cup N_1 = \omega \) and \( N_0 \cap N_1 = B \). Let \( R \) and \( S \) be recursive sets such that \( \bar{M}_0 \subseteq R \), \( \bar{M}_1 \subseteq \bar{R} \), \( \bar{N}_0 \subseteq S \), and \( \bar{N}_1 \subseteq \bar{S} \). (These r.e. sets exist by application of the r.e. reduction principle to the pairs \( M_0, M_1 \) and \( N_0, N_1 \).)

Now observe that \( R \cap A_0 \) and \( R \cap A_1 \) are nonrecursive. For if \( R \cap A_0 \) is
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recursive, then $R - A_0$ is recursive so that $R - A$ is r.e. by the definition of Friedberg splitting. However, $R - A$ is infinite and $A$ is simple. Thus we have that $R \cap A_0$ and $R \cap A_1$ form a nontrivial splitting of $R \cap A$ and similarly $S \cap B_0$ and $S \cap B_1$ form a nontrivial splitting of $S \cap B$. Now within the recursive set $R$, $R \cap A$ is maximal. ($\overline{A} \cap R = \overline{A_0}$ and the latter is cohesive.) Similarly $S \cap B$ is maximal in $S$. Thus by Theorem 3, there is an automorphism $\Phi_0$ of $\mathcal{E}$ restricted to $R$ which maps $R$ to $S$ and $R \cap A_i$ to $S \cap B_i$ for $i = 0, 1$. Likewise, there is an automorphism $\Phi_1$ of the r.e. sets restricted to $R$ which maps $R \cap A_i$ to $S \cap B_i$ for $i = 0, 1$. Piecing together $\Phi_0$ and $\Phi_1$ gives the desired automorphism.

**Corollary 1.** The class of Friedberg splittings of $k$-quasimaximal sets from an orbit for each $k$.

**Proof.** We have show that if $A$ and $B$ are halves of Friedberg splittings of 2-quasimaximal sets then there is a $\Phi$ such that $\Phi(A) = B$. Conversely, if $A$ is half of a Friedberg splitting of a 2-quasimaximal set and $\Phi(A) = B$, then $B$ must be half of a Friedberg splitting of a 2-quasimaximal set since this is an elementary property and so must be preserved under automorphism.

**Corollary 2.** There are four different automorphism types of hemi-2-quasimaximal sets.

**Proof.** Let $M_0, M_1$ be maximal sets such that $M_0 \cup M_1 = \omega$ and $M_0 \cap M_1 = A$. Let $A_0, A_1$ be a notrivial splitting of $A$. Let $R$ be a recursive set such that $\overline{M_0} \subseteq R$ and $\overline{M_1} \subseteq \overline{R}$ and further that all of $R \cap A_0$, $R \cap A_1$, $\overline{R} \cap A_0$, and $\overline{R} \cap A_1$ are infinite. Now one of the following is true:

$$R \cap A_0 \text{ and } \overline{R} \cap A_1 \text{ are nonrecursive, or}$$

(16)

$$R \cap A_1 \text{ and } \overline{R} \cap A_0 \text{ are nonrecursive.}$$

(17)

For if both of the above fail then one of $\overline{M_0}, \overline{M_1}, A_0,$ or $A_1$ is recursive. We will suppose that (16) is true. Of the remaining pair of sets, there are four possibilities:

1. $R \cap A_1$ and $\overline{R} \cap A_0$ are nonrecursive,
2. $R \cap A_1$ and $\overline{R} \cap A_0$ are recursive,
3. $R \cap A_1$ is nonrecursive and $\overline{R} \cap A_0$ is recursive, and
4. $R \cap A_1$ is recursive and $\overline{R} \cap A_0$ is nonrecursive.

All four possibilities can occur; note that (3) and (4) are symmetric.

Case 1. We have already seen in the Theorem that any sets $A_0$ and $A_1$ which arise in this way are of a single orbit.
Case 2. In this case, we see that \( R \cap A_0 \) is maximal in \( R - (R \cap A_1) \) which is a recursive set. Since all such sets are automorphic by Soare's theorem, all such sets \( A_0 \) arising in this way are automorphic. \( A_1 \) is also of this type. This class is distinct from case (1) since \( R - (R \cap A_1) \) witnesses that \( A_0, A_1 \) do not form a Friedberg splitting of \( A \).

Case 3 (and Case 4 by symmetry). In this case, \( S = \bar{R} - A_0 \) is recursive and so \( S \cup A_1 = S \cup (A_1 \cap R) \) is nonrecursive. This implies that \( S \cup A_1 \) and \( A_0 \) form a nontrivial splitting of \( M_1 \). Thus \( A_0 \) is hemimaximal. This determines the automorphism type of \( A_0 \) by Theorem 3. This class is distinct from those of cases 1 and 2 since in neither of those cases is \( A_0 \) hemimaximal (this can easily be checked). Now \( A_1 \) is not hemimaximal but it is easy to show again that all such \( A_1 \) are automorphic (by piecing together automorphisms on \( R \) and \( \bar{R} \) as in the Theorem 13). This gives the fourth and final automorphism type.

Summarizing, there are four automorphism types of hemi-2-quasimaximal sets:

1. half of a Friedberg splitting of a 2-quasimaximal set,
2. maximal in an infinite–cofinite recursive set \( R \),
3. hemimaximal, and
4. not hemimaximal but half of a nontrivial splitting of a 2-quasimaximal set for which the other half is hemimaximal.

The above classification for hemi-2-quasimaximal sets can be extended to hemi-\( k \)-quasimaximal sets for \( k > 2 \) without much difficulty (but much detail).

5. Further Remarks and Open Questions

Theorem 3 justified our hope of finding orbits through splittings. While Theorem 13 gave a negative result along these lines, we still have some hope that Conjecture 2 or some conjecture like it might be true. A test question in this program is the following:

**Question 1.** Let \( A \) and \( B \) be promptly simple and low. Let \( A_0, A_1 \) and \( B_0, B_1 \) be Friedberg splittings of \( A \) and \( B \). Is there \( \Phi \in \text{Aut}(\mathcal{E}) \) such that \( \Phi(A_0) = B_0 \)? (Maass has shown that there is \( \Phi \in \text{Aut}(\mathcal{E}) \) with \( \Phi(A) = B \).)

Note that even Theorem 13 does not refute Conjecture 2 for nonsimple sets since the splittings produced there are not Friedberg splittings.

**Question 2.** What are the automorphism types among Friedberg splittings of a creative set?
The hemimaximal sets form an interesting degree-theoretic class. Left open by the results of Section 2 is

**Question 3.** Are there degrees containing no hemimaximal set which are not low?

Concerning the halfhemimaximal sets, a question which we have been unable to answer is

**Question 4.** Is $K$ halfhemimaximal?

We have shown that creative sets are halfhemi-2-quasimaximal.

We have introduced the properties hemimaximal and halfhemimaximal in this paper because of their use in finding orbits of $\text{Aut}(\mathcal{S})$. However, for various properties $P$, the hemi$P$ and half$P$ sets might prove interesting in their own right. For instance, in contrast with Theorem 11, we have been able to show that although there are nonhalfhemisimple sets, there are degrees containing only halfhemisimple sets.

**REFERENCES**


