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John Ferdinands
Calvin University

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John Ferdinands

Department of Mathematics and Computer Science, Calvin College, Grand Rapids, MI 49546, USA

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Abstract

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A finite CW complex X is said to be *prime* if, given a Hurewicz fibration $F \to E \to B$ with E homotopy equivalent to X, and B and F homotopy equivalent to finite CW complexes, either B or F is contractible. We show that certain 3- and 4-plane complex Grassmannian manifolds are prime.

Keywords: Complex Grassmannian manifold, compact fibering, prime, connectedwise prime, transfer.

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1. Introduction

If G is a compact Lie group and H is a closed subgroup, G/H is known as a homogeneous space. If K is a closed subgroup strictly between H and G, then there is a nontrivial fibering $K/H \to G/H \to G/K$. In [15], Schultz conjectured that if H were maximal, and if some other conditions were imposed, then there were no nontrivial fiberings of G/H.

This conjecture has been verified for several special cases. For example, the even-dimensional projective spaces over the real numbers, complex numbers, quaternions and Cayley numbers have no nontrivial fiberings. (See [2, 6].) In [15], Schultz verified the conjecture for several other homogeneous spaces; in particular, for odd-dimensional quaternionic projective spaces, and for Grassmannian manifolds of 2-planes in complex n-space for $n \geq 5$. This paper tests the conjecture for some 3- and 4-plane complex Grassmannians.

To state our results precisely, we will need the following definitions which are due to Gottlieb. If X is a finite CW complex, a *compact fibering* of X is a Hurewicz fibration $F \to E \to B$ with E homotopy equivalent to X, and B and F homotopic to finite CW complexes. We say that X is *prime* if, given any compact fibering $F \to E \to B$ of X, either B or F is contractible. In addition if X is connected, we say that X is *connectedwise prime* if, given a compact fibering $F \to E \to B$ with F connected, either B or F is contractible.

In this paper the m-plane Grassmannian in complex n-space is denoted by $G_{n,m}(C)$. The following results hold.

Theorem 1.1. *$G_{n,3}(C)$ is prime for $n = 4k+3$, where k is a positive integer.*

Theorem 1.2. *For $n = 8k+4$ and $8k+6$, $G_{n,4}(C)$ is connectedwise prime for all positive integers k, and is prime for all $k \geq 4$.*

Theorem 1.3. *For $n = 9k+3$ and $9k+6$, $G_{n,3}(C)$ is connectedwise prime for all positive integers k. $G_{9k+3,3}(C)$ is prime for all even k, and $G_{9k+6,3}(C)$ is prime for all odd k.*

We shall give a fairly detailed proof of Theorem 1.1. The proofs of Theorems 1.2 and 1.3 are very similar to that of Theorem 1.1, and differ mostly in the details of computation. Accordingly the proof of Theorem 1.2 will be omitted, and that of Theorem 1.3 will be given only for one special case.

2. Proof of Theorem 1.1

We will first prove that $G_{4k+3,3}(C)$ is connectedwise prime for all positive integers k. To show this, we follow the strategy used in [15]. Suppose that $G_{4k+3,3}(C)$ is not connectedwise prime. Then it has a compact fibering $F \xrightarrow{i} E \xrightarrow{p} B$ with neither F nor B being contractible, and with F connected. Since $G_{4k+3,3}(C)$ is a closed oriented manifold, both B and F satisfy a strong form of Poincaré duality (see [13, 18]). We show that the cohomological properties of $G_{4k+3,3}(C)$ imply that either B or F has zero-dimensional cohomology. Since a zero-dimensional Poincaré–Wall complex is contractible [18], we will have a contradiction.

Since $G_{4k+3,3}(C)$ is simply connected, a lemma in [15] implies that F and B are simply connected also. By a result in [9], the compact fibering $F \to E \to B$ is what Halperin calls a *rational fibration.* The rational cohomology spectral sequence collapses [9], and therefore $H^*(B; \mathbb{Q})$ maps injectively into $H^*(E; \mathbb{Q})$.

It is well known [5] that the rational cohomology of $G_{4k+3,3}(C)$, and therefore of E, is a truncated polynomial algebra with generators in dimensions 2, 4 and 6, and relations in dimensions $8k+2$, $8k+4$ and $8k+6$. In other words, $H^*(E; \mathbb{Q}) = \mathbb{Q}[x_2, x_4, x_6]/I$, where x_2, x_4, x_6 are the generators of the polynomial algebra, and I is the ideal generated by the relations. Since $H^*(B; \mathbb{Q})$ maps injectively into

$H^*(E;\mathbb{Q})$, it may be regarded as a subalgebra. Let D be the inverse image of $H^*(B;\mathbb{Q})$ under the natural map $\mathbb{Q}[x_2, x_4, x_6] \to H^*(E, \mathbb{Q})$. Then $H^*(B;\mathbb{Q})$ is isomorphic to D/I, where D is a subalgebra of the free polynomial algebra $\mathbb{Q}[x_2, x_4, x_6]$, and hence D can be written as a polynomial algebra with at most three generators. Since $H^*(E;\mathbb{Q})$ is zero in odd dimensions, this is true of $H^*(B;\mathbb{Q})$ also. So let $H^*(B;\mathbb{Q})$ have generators x, y and z in dimensions $2a$, $2b$ and $2c$ respectively. Then [15] the rational Poincaré polynomials for B and F are given by:

$$P(B)=\frac{(1-t^{8k+2})(1-t^{8k+4})(1-t^{8k+6})}{(1-t^{2a})(1-t^{2b})(1-t^{2c})},$$

$$P(F)=\frac{(1-t^{2a})(1-t^{2b})(1-t^{2c})}{(1-t^{2})(1-t^{4})(1-t^{6})}.$$

For simplicity, let $u=t^2$. Then we get:

$$P(B)=\frac{(1-u^{4k+1})(1-u^{4k+2})(1-u^{4k+3})}{(1-u^{a})(1-u^{b})(1-u^{c})},$$

$$P(F)=\frac{(1-u^{a})(1-u^{b})(1-u^{c})}{(1-u)(1-u^{2})(1-u^{3})}.$$

Since $P(F)$ is a finite polynomial, at least one of a, b and c must be even, say a. By looking at $P(B)$, we see that a must divide $4k+2$, so $a=2m$, where m is odd. We claim that both b and c must be odd. To see this, note that the Euler characteristic of B is given by:

$$\chi(B)=\lim_{u\to 1}\frac{(1-u^{4k+3})(1-u^{4k+2})(1-u^{4k+1})}{(1-u^{a})(1-u^{b})(1-u^{c})}$$

$$=\frac{(4k+3)(4k+2)(4k+1)}{a\cdot b\cdot c}.$$

Since $\chi(B)$ is an integer and a is even, b and c must be odd.

The Poincaré polynomial of $G_{4k+3,3}(C)$ is given by:

$$P(E)=\frac{(1-t^{8k+6})(1-t^{8k+4})(1-t^{8k+2})}{(1-t^{2})(1-t^{4})(1-t^{6})}.$$

(For a reference, see [5].) Putting $u=t^2$, we see that the Euler characteristic of E is given by:

$$\chi(E)=\lim_{u\to 1}\frac{(1-u^{4k+3})(1-u^{4k+2})(1-u^{4k+1})}{(1-u)(1-u^{2})(1-u^{3})}$$

$$=\frac{(4k+3)(4k+2)(4k+1)}{1\cdot 2\cdot 3}.$$

Clearly $\chi(E)$ is odd.

We now use the properties of the Becker-Gottlieb transfer [2, 3]. The transfer is a map $\tau: S^n \wedge B^+ \to S^n \wedge E^+$ for some positive integer n, which has the property that $\tau^* p^*: H^*(B; \mathbb{Z}_2) \to H^*(B; \mathbb{Z}_2)$ is multiplication by $\chi(F)$. Since $\chi(E)$ is odd, and $\chi(E) = \chi(B)\chi(F)$ for compact fibrations (see [16, pp. 491-492]), $\chi(F)$ is odd, and hence p^* is a split monomorphism in $\mathbb{Z}_2$ cohomology. So $H^*(B; \mathbb{Z}_2)$ is a subalgebra of $H^*(E; \mathbb{Z}_2)$.

It is known (see [5]) that $H^*(E; \mathbb{Z})$ is a polynomial algebra on the Chern classes c_1, c_2 and c_3, with relations in dimensions $8k+2$, $8k+4$ and $8k+6$. This implies [11] that $H^*(E; \mathbb{Z}_2)$ is a polynomial algebra on the Stiefel-Whitney classes w_2, w_4 and w_6, with relations which are the images mod 2 of the relations for $H^*(E; \mathbb{Z})$.

We now prove the following important lemma:

Lemma 2.1. *$H^*(B; \mathbb{Z}_2)$ and $H^*(B; \mathbb{Q})$ have the same number of generators, and in the same dimensions. Furthermore, the relations for $H^*(E; \mathbb{Z}_2)$ are polynomials in the images under p^* of the generators of $H^*(B; \mathbb{Z}_2)$.*

Proof. First we prove that $H^*(B; \mathbb{Z})$ has no elements of finite even order. For if u is such an element, then $\tau^* p^*(u) = \chi(F)u \neq 0$, since $\chi(F)$ is odd. So $p^*(u)$ is a nonzero element of finite order, which contradicts the fact that $H^*(E; \mathbb{Z})$ is torsion-free.

Next we see by the Universal Coefficient Theorem that $H^*(B; \mathbb{Z}_2)$ is equal to the free part of $H^*(B; \mathbb{Z})$ tensored with $\mathbb{Z}_2$. This proves the first sentence of the lemma.

By what has been discussed earlier, $H^*(E; \mathbb{Z}_2) = \mathbb{Z}_2[w_2, w_4, w_6]/J$, where J is the ideal generated by the relations for $H^*(E; \mathbb{Z}_2)$. $H^*(B; \mathbb{Z}_2)$ is a subalgebra of $H^*(E; \mathbb{Z}_2)$, and hence, by the same argument as for the case with rational cohomology, $H^*(B; \mathbb{Z}_2)$ is isomorphic to G/J, where G is a free polynomial algebra over $\mathbb{Z}_2$. By the first sentence of this lemma, G has three generators, say $\tilde{x}$, $\tilde{y}$ and $\tilde{z}$. Since $G \supset J$, the relations for $H^*(E; \mathbb{Z}_2)$ must be polynomials in $\tilde{x}$, $\tilde{y}$ and $\tilde{z}$. This completes the proof of the lemma. □

Let the relations for $H^*(E; \mathbb{Z}_2)$ be r_{8k+2}, r_{8k+4} and r_{8k+6}. These are polynomials in w_2, w_4 and w_6, and can be computed as follows (see [5]):

$$r_{8k+2} = w_2\tilde{w}_{8k} + w_4\tilde{w}_{8k-2} + w_6\tilde{w}_{8k-4},$$

$$r_{8k+4} = w_4\tilde{w}_{8k} + w_6\tilde{w}_{8k-2},$$

$$r_{8k+6} = w_6\tilde{w}_{8k},$$

$$(1 + w_2 + w_4 + w_6)(1 + \tilde{w}_2 + \tilde{w}_4 + \cdots) = 1.$$

From the fourth equation we can solve for each $\tilde{w}_i$ in terms of w_2, w_4 and w_6, and thus obtain r_{8k+2}, r_{8k+4} and r_{8k+6} in terms of w_2, w_4 and w_6.

We will need the action of the Steenrod Squares on w_2, w_4 and w_6. These actions are given in the following lemma.

Lemma 2.2. $Sq^2(w_4) = w_2w_4 + w_6$, $Sq^2(w_6) = w_2w_6$ *and* $Sq^4(w_6) = w_4w_6$.

Proof. The results follow immediately from Wu's formula

$$Sq^k(w_m) = w_kw_m + \binom{k-m}{1} w_{k-1}w_{m+1} + \cdots + \binom{k-m}{k} w_0w_{m+k}$$

[11, p. 94]. □

Let K be the ideal of $H^*(E; \mathbb{Z}_2)$ generated by w_2 and w_6. Then we obtain the following lemmas, which are easily proved by induction.

Lemma 2.3. $\tilde{w}_4^i = w_4^i \bmod K$.

Lemma 2.4. $r_{8k+4} = w_4^{2k+1} \bmod K$.

Lemma 2.5. $Sq^4(w_4^i) = w_4^{i+1} \bmod K$, *if i is odd.*

By Lemma 2.1, r_{8k+4} is a polynomial in $\tilde{x}$, $\tilde{y}$ and $\tilde{z}$. Lemma 2.4 implies that at least one of the generators must be equal to a power of w_4 mod K. But only one of these generators is in a dimension divisible by 4, say $\tilde{x}$, and this dimension is equal to $4m$, where m is odd. So $\tilde{x} = w_4^m \bmod K$. Then, by Lemma 2.5, $Sq^4(\tilde{x}) = w_4^{m+1} \bmod K$.

Since $H^*(B; \mathbb{Z}_2)$ is a subalgebra of $H^*(E; \mathbb{Z}_2)$, it has no relations in dimensions less than $8k+2$. There are two cases.

Case 1. $\dim Sq^4(\tilde{x}) < 8k+2$. Since $Sq^4(\tilde{x}) = w_4^{m+1} \bmod K$, $Sq^4(\tilde{x})$ must be nonzero. The Steenrod operations commute with p^*, so $Sq^4(\tilde{x})$ is a polynomial in $\tilde{x}$, $\tilde{y}$ and $\tilde{z}$. But both $\tilde{y}$ and $\tilde{z}$ are zero mod K, so $Sq^4(\tilde{x})$ must be equal to a power of $\tilde{x}$ mod K. For dimensional reasons this implies that $4m$ divides $4m+4$, and hence that $m = 1$. Therefore $\dim \tilde{x} = 4$.

Case 2. $\dim Sq^4(\tilde{x}) \geq 8k+2$. Then $4m+4 \geq 8k+2$. But $4m$ divides $8k+4$, so either $4m = 8k+4$, or $4m \leq \frac{1}{3}(8k+4)$. In the latter case, $4m+4 \leq \frac{1}{3}(8k+4)+4 < 8k+2$ for all $k \geq 1$. So necessarily $4m = 8k+4$. Therefore $\dim \tilde{x} = 8k+4$.

We shall consider each case separately, and show that they both lead to contradictions.

Case 1. $\dim \tilde{x} = 4$.

$\tilde{x}$ is in $H^4(E; \mathbb{Z}_2)$, and hence $\tilde{x} = \alpha w_2^2 + w_4$, where $\alpha = 0$ or 1. (Recall that $\tilde{x}$ is equal to a power of w_4 mod K). Then $Sq^2(\tilde{x}) = w_2w_4 + w_6$, which is nonzero. Therefore $Sq^2(\tilde{x})$ must be a generator for $H^*(B; \mathbb{Z}_2)$. Let $Sq^2(\tilde{x}) = \tilde{y}$.

Now we need the following lemma, which again is easily proved by induction.

Lemma 2.6. $r_{8k+2} = w_2^{4k+1} \bmod L$, *where L is the ideal generated by* w_4 *and* w_6.

Since r_{8k+2} is a polynomial in $\tilde{x}$, $\tilde{y}$ and $\tilde{z}$, at least one of the generators must equal an *odd* power of w_2 mod L. This can only be true for the third generator $\tilde{z}$. Hence $\dim \tilde{z} \leq 8k+2$. Now there are three possibilities.

(1) dim $\tilde{z} \leq \frac{1}{3}(8k+6)$. Then dim $\tilde{z}^2 < 8k+2$, so $\tilde{z}^2 \neq 0$. We have seen that dim $\tilde{z} = 2c$, where c is odd. Note that $c \neq 1$. For, since $G_{4k+3,3}(C)$ is a Kahler manifold of dimension $24k$, the cohomology class c_1 in $H^2(E; \mathbb{Z})$ corresponding to the closed 2-form defined by the Kahler metric satisfies the condition $c_1^{12k} \neq 0$. (See [10].) But $H^2(E; \mathbb{Z}) = \mathbb{Z}$, so the generator in dimension 2 in rational cohomology, and hence also in $\mathbb{Z}_2$ cohomology, must satisfy this condition. If dim $\tilde{z} = 2$, then $\tilde{z} = w_2$. Since $w_2^{12k} \neq 0$, this implies that $\dim(B) = \dim(E)$. Quinn's Formula $\dim(E) = \dim(B) + \dim(F)$ [13] implies that $\dim(F) = 0$, so we have a contradiction.

Since $c > 1$, by the Adem relations [12] we have:

$$\mathrm{Sq}^2\mathrm{Sq}^{2c-2}(\tilde{z}) = \mathrm{Sq}^{2c}(\tilde{z}) = \tilde{z}^2 \neq 0.$$

Hence $\mathrm{Sq}^{2c-2}(\tilde{z}) \neq 0$, so $\mathrm{Sq}^{2c-2}(\tilde{z})$ is a nonzero polynomial in $\tilde{x}$ and $\tilde{y}$.

It is easily checked that $\mathrm{Sq}^2(\tilde{x}^{2m}) = 0$, and $\mathrm{Sq}^2(\tilde{x}^{2m+1}) = \tilde{x}^{2m}\tilde{y}$. Therefore, since $\mathrm{Sq}^{2c-2}(\tilde{z})$ is a polynomial in $\tilde{x}$ and $\tilde{y}$, $\mathrm{Sq}^2\mathrm{Sq}^{2c-2}(\tilde{z})$ is also a polynomial in $\tilde{x}$ and $\tilde{y}$. But $\mathrm{Sq}^2\mathrm{Sq}^{2c-2}(\tilde{z}) = \tilde{z}^2$, which is a contradiction, since there are no relations between the generators in dimensions less than $8k+2$.

(2) dim $\tilde{z} = 8k+2$. We return to rational cohomology. $H^*(B; \mathbb{Q})$ has generators in dimensions 4, 6 and $8k+2$. Therefore the Poincaré polynomials of B and F are given by:

$$P(B) = \frac{(1-u^{4k+2})(1-u^{4k+3})}{(1-u^2)(1-u^3)},$$

$$P(F) = \frac{1-u^{4k+1}}{1-u}.$$

By the Chern-Hirzebruch-Serre Theorem [7], the signatures of E, B and F satisfy the relation $\mathrm{sgn}(E) = \mathrm{sgn}(B)\,\mathrm{sgn}(F)$. It is well known that:

$$\mathrm{sgn}(G_{n,m}(C)) = \binom{[\frac{1}{2}n]}{[\frac{1}{2}m]} \quad \text{if } m(n-m) \text{ is even.}$$

(Compare [14].) This implies that $\mathrm{sgn}(E) = 2k+1$.

Clearly $\mathrm{sgn}(F) = \pm 1$. B has dimension $16k$, so $\mathrm{sgn}(K)$ is at most the dimension of $H^{8k}(B; \mathbb{Q})$ as a rational vector space, which dimension is given by the coefficient of u^{4k} in the Poincaré polynomial of B. We have:

$$P(B) = (1-u^{4k+2})(1-u^{4k+3})(1+u^2+u^4+\cdots)(1+u^3+u^6+\cdots).$$

The coefficient of u^{4k} is the number of ways of partitioning $4k$ into two nonnegative integers of the form $2m$ and $3n$. Since $2m+3n=4k$, we must have $n \leq \frac{4}{3}k$. If k is even, n can take only even values, and if k is odd, n can take only odd values. In either case, there are at most $\frac{2}{3}k+1$ possible values for n. Therefore $\mathrm{sgn}(B) \leq \frac{2}{3}k+1$. Then, by the Chern-Hirzebruch-Serre Theorem, $\mathrm{sgn}(E) = \mathrm{sgn}(B)\,\mathrm{sgn}(F) \leq \frac{2}{3}k+1$. But we have seen that $\mathrm{sgn}(E) = 2k+1$, so we have a contradiction.

(3) $\dim \tilde{z} = 4k+2$. By using rational cohomology and the Chern-Hirzebruch-Serre Theorem, we can obtain a contradiction for $k \geq 2$. The details will be omitted. For $k=1$ we have

$$P(B) = \frac{(1-u^5)(1-u^6)(1-u^7)}{(1-u^2)(1-u^3)(1-u^3)}.$$

But this is not a finite polynomial, as can be seen by putting $u = e^{2\pi i/3}$.

Case 2. $\dim \tilde{x} = 8k+4$. The other generators $\tilde{y}$, $\tilde{z}$ have dimensions $2b$, $2c$ respectively. Assume that $b \leq c$. With rational coefficients we have:

$$P(B) = \frac{(1-u^{4k+1})(1-u^{4k+3})}{(1-u^b)(1-u^c)}.$$

If $b = 4k+1$, then necessarily $c = 4k+3$, which gives $P(B) = 1$, and implies that B is contractible. So we may assume that $b \leq \frac{1}{3}(4k+3)$. This implies that $\dim(\tilde{y}^2) = 4b < 8k+2$, so $\tilde{y}^2 \neq 0$. Therefore

$$\mathrm{Sq}^2\mathrm{Sq}^{2b-2}(\tilde{y}) = \mathrm{Sq}^{2b}(\tilde{y}) = \tilde{y}^2 \neq 0.$$

This means that $\mathrm{Sq}^{2b-2}(\tilde{y}) \neq 0$, so $\mathrm{Sq}^{2b-2}(\tilde{y})$ must be the generator $\tilde{z}$.

Since $H^*(B; \mathbb{Z}_2)$ is a polynomial algebra in $\tilde{x}$, $\tilde{y}$ and $\tilde{z}$, $H^i(B; \mathbb{Z}_2) = 0$ for $2b < i < 4b-2$. Hence Sq^{2b-2} cannot decompose. It can be shown, using the Adem relations, that Sq^i is indecomposable in terms of Steenrod squares if and only if i is a power of 2. (See [12].) It was further shown by Adams in [1] that Sq^{2^n} decomposes via stable secondary cohomology operations for $n \geq 4$. Hence we must have $2b-2 = 2$, 4 or 8. Since b is odd, the only possibilities are $b = 3$ and $b = 5$.

(1) $b = 3$. Then $\tilde{y} = \alpha w_2^4 + \beta w_2 w_4 + \gamma w_6$ for some α, β and γ in $\mathbb{Z}_2$. So we have:

$$\mathrm{Sq}^2(\tilde{y}) = \alpha w_2^3 + (\beta + \gamma) w_2 w_6.$$

But we know that $H^8(B; \mathbb{Z}_2) = 0$, so $\mathrm{Sq}^2(\tilde{y}) = 0$. This implies that $\alpha = 0$ and $\beta = \gamma = 1$. But then $\tilde{y} = w_2 w_4 + w_6 = \mathrm{Sq}^2(w_4)$. p^* is split monic over the algebra of stable cohomology operations [3], and hence if τ is the transfer map, $H^*(E; \mathbb{Z}_2) = \operatorname{im} p^* \oplus \ker \tau^*$, and $\operatorname{im} p^*$ and $\ker \tau^*$ are invariant under the Steenrod squares. Since w_4 is in $\ker \tau^*$, so is $\mathrm{Sq}^2(w_4)$. But we have just seen that $\mathrm{Sq}^2(w_4) = \tilde{y}$, so we have a contradiction.

(2) $b = 5$. Then $\tilde{y}$ has dimension 10, and $\tilde{z} = \mathrm{Sq}^8(\tilde{y})$ has dimension 18. By the Adem relations,

$$\mathrm{Sq}^4\mathrm{Sq}^6(\tilde{y}) + \mathrm{Sq}^8\mathrm{Sq}^2(\tilde{y}) = \mathrm{Sq}^{10}(\tilde{y}) = \tilde{y}^2 \neq 0.$$

But $H^i(B; \mathbb{Z}_2) = 0$ for $10 < i < 18$, so $\mathrm{Sq}^6(\tilde{y})$ and $\mathrm{Sq}^2(\tilde{y})$ must both be zero. Thus we get a contradiction.

This completes the proof that $G_{4k+3,3}(C)$ is connectedwise prime. If $G_{p+q,p}(C)$ with $q > p$, $p \leq 3$ and pq even, is connectedwise prime, it is prime. In [15], Schultz proves this for $p = 2$, using a result from [8] on the nonexistence of fixed-point free self maps for certain complex Grassmannians. The same argument goes through for $p = 3$ also. Therefore $G_{4k+3,3}(C)$ is prime. □

$G_{n,4}(C)$ has odd Euler characteristic for $n = 8k+4$ and $8k+6$, so we again use $\mathbb{Z}_2$ cohomology and the Steenrod Squares to prove it is connectedwise prime. The fact that these manifolds are prime for $k \geq 4$ follows from the result that if $G_{p+q,p}(C)$ with $q > p \geq 4$, pq even and $q \geq 2p^2 - p - 1$ is connectedwise prime, then it is prime. Again, this may be proved by the method used in [15], with the help of a result in [8].

3. Proof of Theorem 1.3

Theorem 1.3. *For $n = 9k+3$ and $9k+6$, $G_{n,3}(C)$ is connectedwise prime for all k. $G_{9k+3,3}(C)$ is prime for all even k, and $G_{9k+6,3}(C)$ is prime for all odd k.*

Proof. The proof will be given only for $n = 9k+3$, and some of the computations will be omitted. The proof for $n = 9k+6$ is very similar.

As before, let us suppose that $F \xrightarrow{i} E \xrightarrow{p} B$ is a nontrivial compact fibering of $G_{9k+3,3}(C)$. With rational coefficients, the Poincaré polynomial of E is given by

$$P(E) = \frac{(1-u^{9k+1})(1-u^{9k+2})(1-u^{9k+3})}{(1-u)(1-u^2)(1-u^3)}.$$

Then it is easily checked that $\chi(E)$ is not divisible by 3, and therefore p^* is a monomorphism in $\mathbb{Z}_3$ cohomology. $H^*(E; \mathbb{Z}_3)$ is a polynomial algebra on generators c_1, c_2 and c_3 in dimensions 2, 4 and 6 respectively, with relations in dimensions $18k+2$, $18k+4$ and $18k+6$. The image of $H^*(B; \mathbb{Z}_3)$ is a subalgebra with generators x, y and z in dimensions $2a$, $2b$ and $2c$ respectively, and the relations for $H^*(E; \mathbb{Z}_3)$ are polynomials in x, y and z.

Let r_{18k+2}, r_{18k+4} and r_{18k+6} be the relations for $H^*(E; \mathbb{Z}_3)$. Let I, J and K be the ideals generated by c_1 and c_2, c_2 and c_3, and c_1 and c_3 respectively. Then the following lemmas hold. Lemmas 3.1 to 3.3 and 3.5 are proved by induction.

Lemma 3.1. $r_{18k+2} = (-1)^{3k} c_3^{3k+1} \bmod I$.

Lemma 3.2. $r_{18k+2} = (-1)^{9k} c_1^{9k+1} \bmod J$.

Lemma 3.3. *If k is even,* $r_{18k+4} = (-1)^{9k/2} c_2^{(9k+2)/2} \bmod K$. *If k is odd,* $r_{18k+2} = (-1)^{9k/2} c_2^{(9k+1)/2} \bmod K$.

Lemma 3.4. *Let P^i denote the Steenrod operations for $p = 3$. Then we have:* $P^1(c_2) = c_1^2 c_2 - c_1 c_3 + c_2^2$, $P^1(c_3) = c_1^2 c_3 + c_2 c_3$, *and* $P^2(c_3) = c_2^2 c_3 + c_1 c_3^2$.

Proof. There is a monomorphism $\rho^* : H^*(\mathrm{BU}(3); \mathbb{Z}_3) \to H^*(\mathrm{BT}^3; \mathbb{Z}_3)$ [4]. Both are free $\mathbb{Z}_3$ polynomial algebras; $H^*(\mathrm{BU}(3); \mathbb{Z}_3)$ has generators c_1, c_2 and c_3 in dimensions 2, 4 and 6 respectively, and $H^*(\mathrm{BT}^3; \mathbb{Z}_3)$ has generators x_1, x_2 and x_3, all in dimension 2. By a result in [4], we have:

$$\rho^*(c_1) = x_1 + x_2 + x_3,$$
$$\rho^*(c_2) = x_1x_2 + x_2x_3 + x_1x_3,$$
$$\rho^*(c_3) = x_1x_2x_3.$$

From the properties of the Steenrod p operations [17], we see that

$$P^1(\rho^*(c_2)) = x_1^3x_2 + x_1x_2^3 + x_2^3x_3 + x_2x_3^3 + x_1^3x_3 + x_1x_3^3 = \rho^*(P^1(c_2)).$$

It may be checked by direct computation that $\rho^*(c_1^2c_2 - c_1c_3 + c_2^2)$ is equal to the above polynomial in the x_i terms. Since ρ^* is a monomorphism, the result for $P^1(c_2)$ follows. $P^1(c_3)$ and $P^2(c_3)$ can be found similarly. □

Lemma 3.5. *Let P^i denote the Steenrod operations for $p = 3$. Then for n not divisible by 3, $P^3(c_3^n) = \pm c_3^{n+2}$ mod I.*

Since r_{18k+6} is a polynomial in x, y and z, Lemma 3.1 implies that at least one of them, say x, is equal to a power of c_3 mod I. But if we use the Poincaré polynomial

$$P(B) = \frac{(1-u^{9k+1})(1-u^{9k+2})(1-u^{9k+3})}{(1-u^a)(1-u^b)(1-u^c)}$$

to compute the Euler characteristic of B, we see that not more than one of a, b and c can be divisible by 3. So only x is a power of c_3 mod I. The dimension of x is $2a$, and hence $a = 3m$ for some m not divisible by 3.

$3m$ must divide $9k+3$. There are three possibilities: $3m \leq \frac{1}{4}(9k+3)$, $3m = \frac{1}{2}(9k+3)$ or $3m = 9k+3$. We shall give the proof only for the first case.

Case 1. $3m \leq \frac{1}{4}(9k+3)$. We have $x = c_3^m$ mod I. By Lemmas 3.4 and 3.5, $P^3(x) = \pm c_3^{m+2}$ mod I.

Now $P^3(x)$ has dimension $6m + 12 \leq \frac{1}{2}(9k+3) + 12 < 18k+2$ for all k. Hence there are no relations between c_1, c_2 and c_3 in this dimension, so $P^3(x) \neq 0$. Since only x is a power of c_3 mod I, $P^3(x)$ must be equal to a power of x mod I. This implies that $6m$ divides $6m+12$, and hence that $m = 1$ or 2. Thus x has dimension 6 or 12.

Now $P^3(x) = \pm c_3^{m+2}$ mod I. Let $2b$ be the smallest dimension in which $H^*(B; \mathbb{Z}_3) \neq 0$. If $\dim(x) = 6$, the possibilities for b are 2 or 3. ($b = 1$ leads to a contradiction as in Section 2, Case 1). If x has dimension 12, b could be 2, 4, 5 or 6. We shall consider only the case where $\dim(x) = 6$ and $b = 2$.

We know that $y = \alpha_1c_1^2 + \alpha_2c_2$ and $x = \beta_1c_1^3 + \beta_2c_1c_2 + c_3$ for some α_i, β_j in $\mathbb{Z}_3$. Then $P^1(x) = c_2c_3 +$ terms containing c_1, and hence is nonzero.

Now suppose $\alpha_2 = 0$. Then $P^1(x)$ cannot be a generator, since Lemma 3.3 implies that some generator is a power of c_2 mod K. But if $P^1(x)$ is not a generator, we must have $P^1(x) = \pm xy$. This is easily seen to be impossible if $\alpha_2 = 0$. So $\alpha_2 \neq 0$, and we may assume that $\alpha_2 = 1$. Then we have $y = \alpha_1c_1^2 + c_2$, $x = \beta_1c_1^3 + \beta_2c_1c_2 + c_3$ and $P^1(y) = -\alpha_1c_1^4 + c_1^2c_2 - c_1c_3 + c_2^2$.

Clearly $P^1(y)$ must be the third generator z, and $P^1(x)=\pm xy$. We claim that $\alpha_1\neq 0$ also. For suppose $\alpha_1=0$. Then the only generator which is nonzero mod J is x. Now Lemma 3.2 implies that r_{18k+2} is equal to a power of x mod J. But dim $x=6$, and 6 does not divide $18k+2$.

Suppose $P^1(x)=xy$. By equating coefficients for this relation, we see that there are two possibilities:

(I) $y=-c_1^2+c_2,\ x=-c_1c_2+c_3$ and $z=c_1^4+c_1^2c_2-c_1c_3+c_2^2$,

(II) $y=c_1^2+c_2,\ x=c_3$ and $z=-c_1^4+c_1^2c_2-c_1c_3+c_2^2$.

From the Poincaré polynomials for E and B, we see that $\dim(E)=54k$ and $\dim(B)=54k-6$. If k is even, the signature formula in [14] shows that $\operatorname{sgn}(E)\neq 0$. But then $\dim(B)$ is not divisible by 4, so the rational cohomology of B vanishes in the middle dimension, and hence $\operatorname{sgn}(B)=0$. Now the Chern-Hirzebruch-Serre Theorem gives a contradiction.

Therefore k must be odd, so $18k+4$ is not divisible by 4. Since r_{18k+4} is a polynomial in x, y and z, and only x has dimension not divisible by 4, r_{18k+4} must have x as a common factor. But if the generators satisfy (II), then this implies that r_{18k+4} has c_3 as a common factor. However we can show by induction that r_{18k+4} has a term $-c_1^{9k}c_2$, so the generators must satisfy (I). So $y=-c_1^2+c_2$, $x=-c_1c_2+c_3$ and $z=c_1^4+c_1^2c_2-c_1c_3+c_2^2$.

Let L be the ideal generated by c_3. When we compute r_{18k+6} as in [5], we see that $r_{18k+6}=0 \bmod L$. Since $y^2=z+c_1c_3$, r_{18k+6} is a polynomial in x and y mod L. Hence, modulo L,

$$r_{18k+6}=\pm(c_1c_2)^{3k+1}+\sum_{p,q}(c_1c_2)^p(-c_1^2+c_2)^q.$$

Then the terms from the summation must yield a term $(c_1c_2)^{3k+1}$. Let $q=3^l m$, where m is not divisible by 3. Then $(-c_1^2+c_2)^q=(-c_1^{2\cdot 3^l}+c_2^{3^l})^m$, since we are using $\mathbb{Z}_3$ coefficients. From the binomial expansion we see that the summation yields a term $(c_1c_2)^{3k+1}$ only if $2\cdot 3^l(m-i)=3^l\cdot i$, or if $i=\frac{2}{3}m$. This is impossible, since 3 does not divide m.

A similar argument works for the case $P^1(x)=-xy$. It turns out that $z=(c_2^2-y^2) \bmod L$, so that, mod L, r_{18k+6} is a polynomial in x, y and c_2^2.

The proofs for the cases $3m=\frac{1}{2}(9k+3)$ and $3m=9k+3$ will be omitted, to save space.

That $G_{9k+3,3}(C)$ is prime if k is even follows from the result, mentioned at the end of Section 2, that if $G_{p+q,p}(C)$ with $q>p$, $p\leqslant 3$ and pq even is connectedwise prime, then it is prime. It also follows that $G_{9k+6,3}(C)$ is prime if k is odd. □

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