5-15-2002

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Topology and its Applications 120 (2002) 1–7
TOPOLOGY AND ITS APPLICATIONS
www.elsevier.com/locate/topol

# The mathematics of T. Benny Rushing

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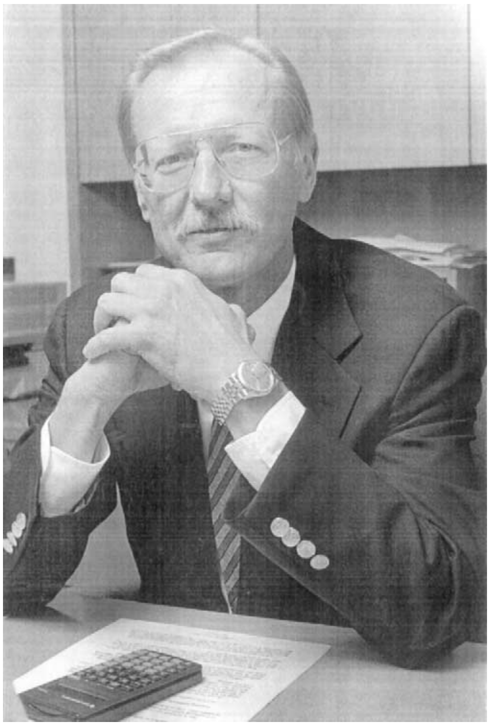

Thomas Benny Rushing, 1941–1998.

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PII: S0166-8641(01)00003-7

This issue is dedicated to the memory of T. Benny Rushing who was our respected colleague and intimate friend. The articles in the issue represent some of the many areas in which he did research and people that he influenced.

We felt it appropriate to include a brief synopsis of his life emphasizing his mathematical contributions as part of the issue. His publications and students stand as poignant reminders of his love of topology and his gifts as a teacher and researcher.

Thomas Benny Rushing was born in Marshville, North Carolina, on October 30, 1941. Even though he spent most of his adult life in other places, he always remained a North Carolinian at heart and he returned there whenever he could. In 1962 Benny married his childhood sweetheart Gail Traywick; together they raised two children, Susan (born 1964) and Tom (born 1967).

Rushing attended Wake Forest University in North Carolina. His first paper [1] was written while he was still an undergraduate there and was published in the *American Mathematical Monthly*. In it he constructs an interesting collection of spaces that are countably compact, but not locally compact or $T_0$. Wake Forest awarded him the bachelors and masters degrees in 1964 and 1965, respectively.

After a short stint in the U.S. Army, Rushing entered graduate school at the University of Georgia where he worked under the direction of J.C. Cantrell. Just two years later, in 1968, Rushing graduated with a Ph.D. in mathematics. Even though he had completed the requirements for his degree, he stayed on at the University of Georgia as an Assistant Professor of Mathematics for the academic year 1968–1969.

Rushing wrote a dissertation on the subject of taming theorems for topological embeddings of polyhedra and manifolds. An embedding $f : P^k \to Q^n$ of a $k$-dimensional polyhedron into an $n$-dimensional piecewise linear (PL) manifold is *tame* if there is an isotopy $e_t : Q \to Q$ such that $e_0$ is the identity and $e_1 \circ f$ is PL; otherwise it is *wild*. The embedding is $\varepsilon$-*tame* if for every $\varepsilon > 0$ it is possible to choose the isotopy $e_t$ in such a way that it moves no point more than a distance $\varepsilon$. Rushing gave conditions under which an embedding is $\varepsilon$-tame. Similar theorems had been proved earlier by Gluck in the trivial range ($n \geqslant 2k + 2$) and by Černavskiĭ in the metastable range ($2n \geqslant 3k + 3$), but Rushing was the first to prove such theorems in codimension three ($n \geqslant k + 3$). The main results were announced in [2], while complete proofs appeared in [5]. The results were generalized and refined in [8] and [9].

In order to state his main theorem it is necessary to make two more definitions. A topological embedding $f : M^k \to Q^n$ of one PL manifold into another is *allowable* if $f^{-1}(\partial Q)$ is a $(k-1)$-dimensional submanifold of $\partial M$. The embedding is *locally flat* if for every point $x$ in the interior of $M$ there exists a neighborhood $U$ of $f(x)$ in $Q$ such that the pair $(U, U \cap f(M))$ is homeomorphic to the pair $(\mathbb{R}^n, \mathbb{R}^k)$.

**Taming Theorem.** *If $f : M^k \to Q^n$, $n \geqslant k + 3$, is an allowable topological embedding of PL manifolds such that $f|f^{-1}(\partial Q)$ and $f|M - f^{-1}(\partial Q)$ are locally flat, then $f$ is $\varepsilon$-tame.*

Several of Rushing's early papers are related to technical questions that arise out of his work on taming theorems. For example, [3] considers the question of whether it is possible to construct an ambient isotopy to be fixed on certain specified polyhedra, while [4] addresses the question of whether a homeomorphism can be realized as the end of an isotopy. In the more technical lemmas in the proof of the taming theorem, the working hypothesis is that the embedding is locally flat on the interior of each simplex of some triangulation of the domain. Considering such hypotheses leads naturally to questions about when an embedding of a cell that is locally flat except possibly on some subset of the boundary is actually flat (addressed in [10]) and whether an embedding that is locally flat on each of several cells is locally flat on the union (addressed in [6]).

In 1969 Rushing was appointed to the faculty of the University of Utah and he remained a member of that faculty until his death in 1998. It was during his first eight years there that he supervised most of the Ph.D. dissertations listed in Appendix B. He was known as an engaging teacher of both undergraduate and graduate students. His offbeat sense of fun and unending curiosity made him a favorite of both students and colleagues. Together with Glaser he organized the highly successful 1974 Geometric Topology Conference in Park City and edited the proceedings of that conference [16].

During his early years at Utah Rushing continued to actively pursue the research interests described above. He constructed [11] embeddings of cells in every dimension and codimension that are wild at every point. At the same time he was devoting a great deal of energy to writing his book "Topological Embeddings" [14]. The book was designed to make the subject of topological embeddings accessible to graduate students. It is written in a highly geometric style that is typical of Rushing's approach to all of mathematics and is lavishly illustrated with numerous interesting schematic pictures. Many of the results included in the book are proved using engulfing. Rushing presents an overview and unified treatment of the various engulfing techniques that were then being used by various authors. In particular, he provides separate treatments of engulfing theorems of both the Stallings and Zeeman types. He also includes a careful treatment of infinite engulfing (see also [13]). The book culminates in a chapter that includes Rushing's taming results as well as an exposition of the work of Černavskiĭ on flattening embeddings of cell pairs.

Around 1974 Rushing became interested in two new research topics: weak flatness and shape theory. An embedding $f : S^k \to S^n$ is *weakly flat* if the complement of $f(S^k)$ in $S^n$ is homeomorphic to the complement of the standard $k$-sphere in $S^n$. Theorems that give homotopy conditions under which an embedding of a codimension three sphere is weakly flat had been proved by Duvall. In collaboration with Hollingsworth, Rushing extended such characterizations to codimension two [20]. A little later he and Daverman gave interesting applications of the codimension two weak flatness theorem [19]. At about that time many topologists were becoming interested in Borsuk's theory of shape. Rushing recognized that the ideas of shape theory could be used to formulate a type of converse to the weak flatness theorem. In particular, he had the insight that if a compactum in $S^n$ has the same complement as does the $k$-sphere, then it must have the same shape as the $k$-sphere. This allowed him to prove a "complement theorem", which is a very geometric type of duality theorem. The following is an example of the type of theorem he proved.

The condition "globally 1-alg" is a technical condition on the fundamental group of the end of the complement of the embedding.

**Complement Theorem.** *Let $X$ be a compact subset of $S^n$, $n \geqslant 5$, that is globally* 1*-alg and let $k$ be an integer $\leqslant n-3$. Then $X$ has the shape of $S^k$ if and only if $S^n - X$ is homeomorphic with $S^n - S^k$.*

Rushing announced the theorem above in [17] and the proof appeared in [21]. Other authors had proved complement theorems before this one, but those earlier theorems either required the ambient manifold to be infinite dimensional or the compactum to have dimension in the trivial range. Rushing, in collaboration with Hollingsworth, proved a trivial range complement theorem of his own [18]. The main difference between the Hollingsworth–Rushing theorem and earlier trivial range complement theorems is that they replaced the hypotheses used in the earlier statements by ones that are more appropriate to the shape theoretic setting.

Rushing spent the 1975–1976 academic year as an exchange scientist at the University of Zagreb where he began a collaboration with Sibe Mardešić. Together they introduced the notion of a shape fibration. Earlier Coram and Duvall had introduced approximate fibrations and these had proved to be quite useful in shape theory. But approximate fibrations have some drawbacks, principally the fact that the definition only makes sense for maps of ANRs and that there are no pull-backs in the theory. Mardešić and Rushing worked out a theory of shape fibrations in [22] and proved in that paper and in [23] that shape fibrations have all the properties needed to make them the natural generalization of fibration to the shape category. Rushing also wrote an expository paper [25] that explained the relationships between cell-like maps, approximate fibrations, and shape fibrations. Mardešić and Rushing gave further generalizations of the notion of a shape fibration in [26].

Rushing had his first bout with cancer in 1978. Thanks to chemotherapy treatments that were quite experimental at the time, he was able to win this first battle with cancer and he enjoyed good health for the next 15 years. After 1978 Rushing devoted more of his time and energy to administrative tasks and he served the mathematical community in many important ways. He was chair of the University of Utah Mathematics Department from 1985–1988 and from 1991–1993. In that position he played a key role in building the department into the nationally acclaimed department that it is today. In 1993 he was appointed Dean of the College of Science at the University of Utah. Shortly after his appointment as dean he once again began to experience health problems. The official publication of the University of Utah College of Science makes the following comment on Rushing's service as dean: "[Rushing] served effectively . . . with selfless devotion and remarkable energy despite the illness that ultimately took his life". [1] He retired from his position as dean in 1997.

During the 1980s, Rushing worked on two major research projects. The first involved a study of the relationship between cell-like maps and disk-bundle projections. Siebenmann

[1] *Notebook*, Vol. VI, Spring 1999, p. 20.

proved that every cell-like map between closed manifolds can be approximated by homeomorphisms; this is a codimension zero theorem and Rushing was seeking a generalization to other codimensions. In that generality the approximations must be by disk-bundle projections rather than by homeomorphisms. His main theorem [30] is the following.

**Approximation Theorem.** *If $p : M^n \to A$ is a cell-like map from a compact $n$-manifold (possibly with boundary) onto an ANR and $\varepsilon > 0$, then there exists a compact ANR $M' \supset M$, a retraction $r : M' \to M$, and an $(n+1)$-disk-bundle map $p' : M' \to A$ which extends $p$ such that* $\mathrm{dist}(p', pr) < \varepsilon$.

Rushing proved a partial converse in [31] and he and Montejano generalized the theorem from cell-like maps to $\alpha$-equivalences in [32]. In an unrelated paper [34] he and Sher constructed an example of a cellular set that is the wedge of two spaces, one of which is not cellular.

Rushing's last paper [35] studies relationships between wild embeddings and fractals. He first observes that for every real number $s$ in the range $[0, n]$ there exist tame Cantor sets in $\mathbb{R}^n$ whose Hausdorff dimension is $s$, but that a wild Cantor set in $\mathbb{R}^n$ must have Hausdorff dimension at least $n-2$. He then proves that for every $s \in [n-2, n]$ there exists a wild Cantor set in $\mathbb{R}^n$ of Hausdorff dimension $s$. Furthermore, if $k$ is an integer in the range $1 \leqslant k \leqslant s$ and $k \neq n$ then there exist everywhere wild $k$-spheres and $k$-cells in $\mathbb{R}^n$ of Hausdorff dimension $s$.

During the last few years of his life Rushing suffered declining health. He experienced heart problems shortly after his appointment as dean. He recovered, but was weaker than before. Several years later he developed a lymphoma which in turn led to a variety of complications. Through all this Rushing maintained high spirits and never lost his gentle southern sense of humor. He fought hard, but he passed away on August 29, 1998. He is greatly missed by all his colleagues and friends.

## Appendix A. Publications of T.B. Rushing


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## Appendix B. Ph.D. Dissertations written under the direction of T.B. Rushing

Orville L. Bierman, "Monotone Union Properties" (1971).
Frederic O. Benson, "Flattening Criteria for Embeddings" (1974).
Vo Thanh Liem, "Embeddings of Shape Classes of Closed Manifolds" (1975).
Gerard A. Venema, "Weak Flatness for Shape Classes" (1975).
Thomas C. McMillan, "Cell-like Maps which are Shape Fibrations" (1977).
Allen Matsumoto, "Obstruction Theory for Shape Fibrations" (1977).
Luis Montejano, "$\beta$-Homotopy Equivalences have $\alpha$-Cross Sections" (1980).
Jorge Martin, "On inverse limits of bundle maps" (1981).