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A homotopy equivalence that is not homotopic to a topological embedding

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Abstract

An open subset W of S^n, $n \geqslant 6$ or $n = 4$, and a homotopy equivalence $f : S^2 \times S^{n-4} \to W$ are constructed having the property that f is not homotopic to any topological embedding.

Keywords: Compact subset; Homotopy equivalence; Topological embedding

AMS classification: 57N13; 57N15; 57N35; 54D30; 55P10; 55P55

1. Introduction

In this paper we construct examples of open manifolds that have unusual compactness properties. The precise properties of the examples are spelled out in the following theorem.

Theorem 1.1. *There exists an open subset W^n of S^n, $n \geqslant 6$, such that W^n has the homotopy type of $S^2 \times S^{n-4}$, but there is no compact subset X^n contained in W^n such that the inclusion $X^n \hookrightarrow W^n$ is a Čech equivalence. There exists an open subset W^4 of S^4 such that W^4 has the homotopy type of S^2, but there is no compact subset X^4 contained in W^4 such that the inclusion $X^4 \hookrightarrow W^4$ is a Čech equivalence.*

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When we say that $X \hookrightarrow W$ is a *Čech equivalence* we mean that the inclusion map induces isomorphisms on the Čech homology and cohomology groups (with $\mathbb{Z}$ coefficients).

One consequence of Theorem 1.1 is the fact that there is a homotopy equivalence between piecewise linear manifolds that is not homotopic to an embedding.

Corollary 1.2. *There is a homotopy equivalence $f : S^2 \times S^m \to W^{m+4}$, $m \geqslant 2$, such that f is not homotopic to any topological embedding.*

The corollary contrasts sharply with what happens in the compact setting. In that setting the following problem has been studied extensively: If $f : M^{n-2} \to W^n$ is a homotopy equivalence from a closed, orientable PL $(n-2)$-manifold M to a compact, orientable n-dimensional PL manifold (with boundary) W, then is f homotopic to a locally flat PL embedding? Cappell and Shaneson [1–3], Kato [4], Kato and Matsumoto [5], and Matsumoto [9–12] have developed a codimension two ambient surgery theory that answers this question in case $n \geqslant 6$. There is an obstruction that vanishes if and only if f is homotopic to a locally flat PL embedding. One consequence of the theory is the fact that for n odd the obstruction group is the same as the ordinary surgery obstruction group. It follows that, in odd dimensions, a (possibly nonlocally flat) PL embedding always exists [2, Theorem 6.1]. Corollary 1.2 shows that this fails in case W is not compact.

It should also be noted that the corollary illustrates a distinct difference between codimension two and codimensions greater than two. For $k \geqslant 3$, every homotopy equivalence from a closed PL n-manifold to a PL $(n+k)$-manifold is homotopic to an embedding. This is true whether or not the target manifold is compact—see the Introduction to [18] for a discussion of this.

There have previously been topological nonembedding results in dimension four ([7] and [17]), but none that we are aware of in higher dimensions.

Another way to view Theorem 1.1 is in terms of compact cores. We say that a compact submanifold N of W is a *core* of W if the inclusion map $N \hookrightarrow W$ is a homotopy equivalence. In this paper we are allowing the subset to be an arbitrary compactum; in that generality it makes sense to define core in terms of shape equivalence. Thus we define a compact subset X of W to be a *generalized core* of W if the inclusion map $X \hookrightarrow W$ is a shape equivalence. Theorem 1.1 asserts that a generalized core may fail to exist even if the manifold has the homotopy type of a finite complex.

Corollary 1.3. *There is an open subset W^n of S^n, $n \geqslant 6$, such that W^n has the homotopy type of a finite complex but W^n contains no generalized core.*

Historical note. Matsumoto proved in 1978 [13] that Kawauchi's Theorem [6] could be generalized to the topological setting. In the summer of 1996 the other two authors rediscovered Matsumoto's 1978 notes and saw how to simplify the argument and extend the techniques to higher dimensions.

2. Background and notation

Since the proof given here builds on that in [18], we will assume that the reader is familiar with the notation and techniques of [18]. In particular, all the notation associated with the construction of W in [18, Section 2] will be assumed. Also the definitions of the Alexander polynomial $A(X,\gamma;t)$ and the Kawauchi invariant $k(X,\gamma)$ in [18, Section 3] and all the lemmas of [18, Section 3] will be assumed. So it is essential that Sections 2 and 3 of [18] be read before this paper. The remainder of this paper will substitute for the proofs found in [18, Section 4].

Most homology and cohomology groups will have coefficients in $\mathbb{Z}$. If no coefficient group is specified, coefficients in $\mathbb{Z}$ are to be assumed. Occasionally we will require rational coefficients; in those cases the coefficient group $\mathbb{Q}$ will be specified.

Readers who are unfamiliar with shape theory should consult [8] for the definitions needed in order to understand the statement of Corollary 1.3. Since the Čech homology and cohomology groups are shape invariants [8, Chapter II], Corollary 1.3 follows immediately from Theorem 1.1.

In the next four sections we will give the details of the proof of Theorem 1.1 in the special case $n = 4$. In the final section we will indicate the modifications that are necessary to prove the high dimensional cases of the theorem.

3. Neighborhoods of X

Suppose X is a compact subset of the 4-manifold W and that $X \hookrightarrow W$ is a Čech equivalence. Compactness implies that there exists an i such that $X \subset \operatorname{Int} W(L_i)$. This i will be fixed for the remainder of the proof. The hypothesis that

$$\check{H}_*(X) \cong \check{H}_*(W) \cong \check{H}_*(S^2)$$

implies that there exists a sequence of connected neighborhoods $N_1, N_2, \ldots$ of X in $\operatorname{Int} W(L_i)$ such that the following conditions are satisfied.

(3.1) $N_{j+1} \subset \operatorname{Int} N_j$ for each $j \geqslant 1$.

(3.2) $\bigcap_{j=1}^{\infty} N_j = X$.

(3.3) The inclusion induced homomorphism $H_k(N_{j+1}) \to H_k(N_j)$ is zero for $k \neq 0$ or 2.

(3.4) If $\alpha_j : H_2(N_{j+1}) \to H_2(N_j)$ denotes the inclusion induced homomorphism, then $\operatorname{im}\alpha_j \cong \mathbb{Z}$.

The fact that the inclusion map $X \hookrightarrow W$ is a Čech homology equivalence allows us to impose one more requirement on the neighborhoods.

(3.5) If $\beta_j : H_2(N_j) \to H_2(W)$ denotes the inclusion induced homomorphism, then $\beta_j | \operatorname{im}\alpha_j : \operatorname{im}\alpha_j \to H_2(W)$ is an isomorphism.

We will use α_j' and β_j' to denote the inclusion induced homomorphisms $\alpha_j' : H_1(N_{j+1} - X) \to H_1(N_j - X)$ and $\beta_j' : H_1(N_j - X) \to H_1(W - X)$. Recall that $W(L_i)$ is constructed from B^4 by attaching a 1-handle and a 2-handle. Let us denote the boundary of the cocore of the 2-handle by b_i. Note that b_i is a loop on $\partial W(L_i) \subset W - X$ and that b_i bounds a disk $c_i \subset W(L_i)$. The pair (c_i, b_i) represents a generator of $H_2(W(L_i), \partial W(L_i)) \cong \mathbb{Z}$.

Lemma 3.1. *If α'_j and β'_j are as above, then $\operatorname{im}\alpha'_j \cong \mathbb{Z}$ and $\beta'_j|\operatorname{im}\alpha'_j : \operatorname{im}\alpha'_j \to H_1(W-X)$ is an isomorphism. Moreover, b_i represents a generator of $H_1(W - X)$.*

Proof. Since N_j is a subset of S^4 we see that the inclusion induced homomorphism $H_2(N_j - X) \to H_2(N_j)$ is onto. Thus the long exact sequence of the pair $(N_j, N_j - X)$ shows that $H_2(N_j) \to H_2(N_j, N_j - X)$ is the zero homomorphism. Now consider the following commutative diagram.

$$\begin{tikzcd}
H_2(N_{j+1}) \arrow[r, "0"] \arrow[d] & H_2(N_{j+1}, N_{j+1} - X) \arrow[r, "\partial_{j+1}"] \arrow[d, "\cong"'] & H_1(N_{j+1} - X) \arrow[r] \arrow[d, "\alpha'_j"] & H_1(N_{j+1}) \arrow[d, "0"] \\
H_2(N_j) \arrow[r, "0"] \arrow[d] & H_2(N_j, N_j - X) \arrow[r, "\partial_j"] \arrow[d, "\cong"'] & H_1(N_j - X) \arrow[r] \arrow[d, "\beta'_j"] & H_1(N_j) \arrow[d] \\
H_2(W) \arrow[r, "0"] & H_2(W, W - X) \arrow[r, "\partial", "\cong"'] & H_1(W - X) \arrow[r] & H_1(W) = 0
\end{tikzcd}$$

The vertical arrows in the second column are isomorphisms by excision. By Alexander duality, each group in the second column is isomorphic to $\check{H}^2(X) \cong \mathbb{Z}$. An easy diagram chasing argument shows that $\operatorname{im}\alpha'_j = \operatorname{im}\partial_j \cong \mathbb{Z}$. Essentially the same argument shows that $\operatorname{im}\beta'_j \circ \alpha'_j = \operatorname{im}\partial = H_1(W - X) \cong \mathbb{Z}$. Since every onto homomorphism $\mathbb{Z} \to \mathbb{Z}$ is an isomorphism, we have that $\beta'_j|\operatorname{im}\alpha'_j : \operatorname{im}\alpha'_j \to H_1(W - X)$ is an isomorphism.

Since $X \hookrightarrow W$ and $W(L_i) \hookrightarrow W$ induce isomorphisms on $\check{H}^2$, $X \hookrightarrow W(L_i)$ does as well. It follows that the horizontal arrow in the lower left corner of the following diagram is an isomorphism.

$$\begin{tikzcd}
H_2(W(L_i), \partial W(L_i)) \arrow[r] \arrow[d, "\cong"'] & H_2(W(L_i), W(L_i) - X) \arrow[r, "\cong"] \arrow[d, "\cong"] & H_2(W, W - X) \arrow[d, "\cong"] \\
H^2(W(L_i)) \arrow[r, "\cong"] & \check{H}^2(X) \arrow[r, equal] & \check{H}^2(X)
\end{tikzcd}$$

The second arrow in the top row is an isomorphism by excision. The vertical arrows are the Alexander and Poincaré duality isomorphisms. The diagram commutes by naturality of duality. Since (c_i, b_i) represents a generator of $H_2(W(L_i), \partial W(L_i))$, it also represents a generator of $H_2(W, W-X)$. But $\partial : H_2(W, W-X) \to H_1(W-X)$ is an isomorphism, so b_i represents a generator of $H_1(W - X)$. $\square$

4. Neighborhoods of C

We now turn our attention to $M(L_i)$. Recall that $M(L_i)$ is the manifold constructed by attaching two 2-handles to B^4 along the components of L_i using zero framing. Thus

$M(L_i) \supset W(L_i)$ and $M(L_i) - \operatorname{Int} W(L_i) \cong S^2 \times B^2$. Let us use Σ to denote the core, $S^2 \times \{0\}$, of $M(L_i) - \operatorname{Int} W(L_i)$. In order to simplify the notation, let us use M to denote $M(L_i)$.

Let A be a PL ray in M that starts at a point of Σ and converges to X. Choose A in such a way that $A \cap \Sigma$ consists of one point and $A \cap X = \emptyset$. Further, choose A so that, for each j, A intersects ∂N_j transversely in exactly one point. In this way we form a compact, connected set $C = X \cup A \cup \Sigma$. By taking the union of N_j with a regular neighborhood of $(A - \operatorname{Int} N_j) \cup \Sigma$ we can form a connected neighborhood P_j of C in M. The sequence of neighborhoods $\{P_j\}$ satisfies the following conditions.

(4.1) $P_{j+1} \subset \operatorname{Int} P_j$ for each $j \geqslant 1$.

(4.2) $\bigcap_{j=1}^{\infty} P_j = C$.

(4.3) The inclusion induced homomorphism $H_k(P_{j+1}) \to H_k(P_j)$ is zero for $k \neq 0$ or 2.

(4.4) If $\phi_j : H_2(P_{j+1}) \to H_2(P_j)$ denotes the inclusion induced homomorphism, then $\operatorname{im} \phi_j \cong \mathbb{Z} \oplus \mathbb{Z}$.

(4.5) If $\psi_j : H_2(P_j) \to H_2(M)$ denotes the inclusion induced homomorphism, then $\psi_j | \operatorname{im} \phi_j : \operatorname{im} \phi_j \to H_2(M)$ is an isomorphism.

The last three conditions are achieved with the aid of a Mayer–Vietoris sequence.

Lemma 4.1. *For every j and for every k, the inclusion induced homomorphism $H_k(M, P_{j+1}) \to H_k(M, P_j)$ is zero.*

Proof. In case $k \neq 2$ or 3, the conclusion follows immediately from the following commutative diagram.

$$\begin{tikzcd}
0 = H_k(M) \arrow[r] \arrow[d, equal] & H_k(M, P_{j+1}) \arrow[r] \arrow[d] & \widetilde{H}_{k-1}(P_{j+1}) \arrow[d, "0"] \\
0 = H_k(M) \arrow[r] & H_k(M, P_j) \arrow[r] & \widetilde{H}_{k-1}(P_j)
\end{tikzcd}$$

In case $k = 2$, the argument is nearly the same but we must extend the diagram one place to the left. We use the fact that ψ_j is onto, by Property 4.5 above.

$$\begin{tikzcd}
H_2(P_{j+1}) \arrow[r, "\psi_{j+1}"] \arrow[d] & H_2(M) \arrow[r] \arrow[d, equal] & H_2(M, P_{j+1}) \arrow[r] \arrow[d] & H_1(P_{j+1}) \arrow[d, "0"] \\
H_2(P_j) \arrow[r, "\psi_j"] & H_2(M) \arrow[r, "0"] & H_2(M, P_j) \arrow[r, "1\text{-}1"] & H_1(P_j)
\end{tikzcd}$$

Finally, in case $k = 3$, the argument is a little more delicate. Let us use θ_j to denote the inclusion induced homomorphism $\theta_j : H_3(M, P_{j+1}) \to H_3(M, P_j)$. We have the following commutative diagram.

$$\begin{tikzcd}
0 = H_3(M) \arrow[r] \arrow[d, equal] & H_3(M, P_{j+1}) \arrow[r, "\partial_{j+1}"] \arrow[d, "\theta_j"'] & H_2(P_{j+1}) \arrow[r, "\psi_{j+1}"] \arrow[d, "\phi_j"] & H_2(M) \arrow[d, equal] \\
0 = H_3(M) \arrow[r] & H_3(M, P_j) \arrow[r, "\partial_j"] & H_2(P_j) \arrow[r, "\psi_j"] & H_2(M)
\end{tikzcd}$$

Let $x \in H_3(M, P_{j+1})$. Then $\psi_j \partial_j \theta_j(x) = 0$ and so $\psi_j \phi_j \partial_{j+1}(x) = 0$. But Property 4.5 implies that $\psi_j | \operatorname{im} \phi_j$ is monic. Hence $\phi_j \partial_{j+1}(x) = 0$. By commutativity of the diagram we have $\partial_j \theta_j(x) = 0$. This implies that $\theta_j(x) = 0$ since exactness of the bottom row implies that ∂_j is a monomorphism. □

Lemma 4.2. $H_*(M - C, \partial M) = 0$.

Proof. Let M_0 denote the manifold obtained from M by deleting a small open collar on ∂M. By Alexander duality we have

$$H_k(M - C, \partial M) \cong H_k(M - C, M - M_0) \cong \check{H}^{4-k}(M_0, C) \cong \check{H}^{4-k}(M, C).$$

Now the definition of Čech cohomology gives

$$\check{H}^{4-k}(M, C) = \lim_{j \to \infty} H^{4-k}(M, P_j).$$

Hence the conclusion follows from Lemma 4.1 and the Universal Coefficient Theorem for Cohomology (see [16, Theorem 53.1], for example). □

5. The infinite cyclic cover of $M - C$

The next step in the proof is to construct an infinite cyclic cover of $M - C$. Recall [18] that $H_1(\partial M; \mathbb{Z}) \cong \mathbb{Z} \oplus \mathbb{Z}$. We use $\mathbb{J}$ to denote the multiplicative infinite cyclic group with generator t. Define $\gamma : \pi_1(\partial M) \to \mathbb{J}$ by sending each of the two generators of $H_1(\partial M; \mathbb{Z})$ to a generator of $\mathbb{J}$. (In order for γ to be uniquely defined we must first choose an orientation for L_i. This choice should be made as in [14].) Lemma 4.2 allows γ to be extended to $\pi_1(M - C)$. Let

$$p : \widetilde{M - C} \to M - C$$

denote the associated infinite cyclic cover.

We now begin to use rational coefficients in our homology groups. The symbol Λ is used to denote the group ring, $\mathbb{Q}[\mathbb{J}]$, which consists of all Laurent polynomials in t with rational coefficients. For each pair of polyhedra (K, L) in $M - C$ we use the notation $H_k(K, L; \Lambda)$ as a shorthand for the homology group $H_k(p^{-1}(K), p^{-1}(L); \mathbb{Q})$. Of course $H_k(K, L; \Lambda)$ has a natural Λ-module structure.

As in [18], we use the same letter t to denote a generator of $\mathbb{J}$, the associated deck transformation of $\widetilde{M - C}$, and the homomorphisms it induces on the homology groups of $\widetilde{M - C}$. Thus a Laurent polynomial in t can be thought of as a homomorphism on homology groups. The transformation induced by the polynomial $t - 1$ is of special interest to us because for each pair of polyhedra (K, L) in $M - C$ we have a long exact sequence of the following form.

$$\cdots \to H_{k+1}(K, L; \mathbb{Q}) \to H_k(K, L; \Lambda) \xrightarrow{t-1} H_k(K, L; \Lambda) \xrightarrow{p_*} H_k(K, L; \mathbb{Q}) \to \cdots.$$

We will refer to this sequence as the *Milnor sequence* of the pair (K, L) because of the important role it plays in the paper [15], where many of the techniques used in this paper have their origin. (This sequence can also be viewed as a special case of the Wang sequence.)

Lemma 5.1. *For each $j \geqslant 1$, the inclusion induced homomorphism*

$$a_j : H_1(N_{j+1} - C; \Lambda) \to H_1(N_j - C; \Lambda)$$

satisfies

$$a_j\big(H_1(N_{j+1} - C; \Lambda)\big) \subset (t-1)\big(H_1(N_j - C; \Lambda)\big).$$

Proof. Note that the only difference between $C \cap N_j$ and $X \cap N_j$ is $A \cap N_j$. But $A \cap N_j$ is a 1-dimensional polyhedron, so removing it from the 4-dimensional manifold N_j has no effect on the first homology group. It therefore follows from Lemma 3.1 that if α_j'' denotes the inclusion induced homomorphism $\alpha_j'' : H_1(N_{j+1} - C; \mathbb{Q}) \to H_1(N_j - C; \mathbb{Q})$, then $\operatorname{im} \alpha_j'' \cong \mathbb{Q}$. It also follows from the last statement in Lemma 3.1 that $p^{-1}(N_j - C)$ is connected for each j and so $H_0(N_j - C; \Lambda) = H_0(p^{-1}(N_j - C); \mathbb{Q}) \cong \mathbb{Q}$. Consider the following commutative diagram in which each row is a portion of a Milnor sequence. (In order to keep the lengths of the rows in the diagram down to a manageable size we use $N_J' = N_j - C$ and $N_{j+1}' = N_{j+1} - C$.)

$$\begin{tikzcd}
H_1(N_{j+1}'; \Lambda) \arrow[r, "t-1"] \arrow[d, "a_j"] & H_1(N_{j+1}'; \Lambda) \arrow[r, "p_{j+1}"] \arrow[d, "a_j"] & H_1(N_{j+1}'; \mathbb{Q}) \arrow[r, "\delta_{j+1}"] \arrow[d, "\alpha_j''"] & H_0(N_{j+1}'; \Lambda) \arrow[r, "t-1", "0"'] \arrow[d, "\cong"] & H_0(N_{j+1}'; \Lambda) \arrow[d, "\cong"] \\
H_1(N_j'; \Lambda) \arrow[r, "t-1"] & H_1(N_j'; \Lambda) \arrow[r, "p_j"] & H_1(N_j'; \mathbb{Q}) \arrow[r, "\delta_j"] & H_0(N_j'; \Lambda) \arrow[r, "t-1", "0"'] & H_0(N_j'; \Lambda)
\end{tikzcd}$$

Exactness of the top row implies that δ_{j+1} is an epimorphism. Thus $\delta_j \alpha_j''$ is epic and so $\delta_j | \operatorname{im} \alpha_j''$ is an epimorphism from $\operatorname{im} \alpha_j''$ to $H_0(N_j - C; \Lambda)$. By the previous paragraph, each of these groups is isomorphic to $\mathbb{Q}$. Since the only epimorphisms from $\mathbb{Q}$ to $\mathbb{Q}$ are isomorphisms, we see that $\delta_j | \operatorname{im} \alpha_j''$ is a monomorphism. Now the composition $\delta_j p_j a_j$ is zero, so $\delta_j \alpha_j'' p_{j+1} = 0$. The previous two sentences together imply that $\alpha_j'' p_{j+1} = 0$. Thus $p_j a_j = 0$ and so $\operatorname{im} a_j \subset \ker p_j = \operatorname{im}(t-1)$. $\square$

Lemma 5.2. *The inclusion induced homomorphisms*

$$b_j : H_1(P_{j+1} - C; \Lambda) \to H_1(P_j - C; \Lambda)$$

and

$$c_j : H_1(M - C, P_{j+1} - C; \Lambda) \to H_1(M - C, P_j - C; \Lambda)$$

satisfy

$$b_j\big(H_1(P_{j+1} - C; \Lambda)\big) \subset (t-1)\big(H_1(P_j - C; \Lambda)\big)$$

and

$$c_j\big(H_1(M - C, P_{j+1} - C; \Lambda)\big) \subset (t-1)\big(H_1(M - C, P_j - C; \Lambda)\big).$$

Proof. The fact about b_j follows from Lemma 5.1 and a Mayer–Vietoris sequence argument. By Lemma 4.1 and excision, the inclusion induced homomorphism $H_1(M - C, P_{j+1} - C; \mathbb{Q}) \to H_1(M - C, P_j - C; \mathbb{Q})$ is zero. Thus the second part of the lemma can be seen from the following commutative diagram in which each row is a Milnor sequence.

$$\begin{tikzcd}
H_1(M-C,P_{j+1}-C,\Lambda) \arrow[r, "t-1"] \arrow[d] & H_1(M-C,P_{j+1}-C,\Lambda) \arrow[r] \arrow[d, "c_j"] & H_1(M-C,P_{j+1}-C;\mathbb{Q}) \arrow[d, "0"] \\
H_1(M-C,P_j-C,\Lambda) \arrow[r, "t-1"] & H_1(M-C,P_j-C,\Lambda) \arrow[r] & H_1(M-C,P_j-C;\mathbb{Q})
\end{tikzcd}$$

6. Proof of Theorem 1.1 in case $n = 4$

We are assuming that the compactum X exists and wish to derive a contradiction from this assumption. In view of Lemma 4.2, we may apply [18, Lemma 3.5] to the pair $(M - C, \partial M)$. Thus $H_1(M - C; \Lambda)_{(t-1)} \cong H_1(\partial M; \Lambda)_{(t-1)}$. In particular, $H_1(M - C; \Lambda)_{(t-1)}$ is finitely generated as a Λ-module. As computed in [14], $k(\partial M, \gamma|\partial M) = 2i$ and so $H_1(\partial M; \Lambda)_{(t-1)} \neq \{0\}$. We will derive our contradiction by proving that $H_1(M - C; \Lambda)_{(t-1)} = \{0\}$. By [18, Lemma 3.2], it suffices to show that $(t-1): H_1(M-C;\Lambda) \to H_1(M - C; \Lambda)$ is onto. This follows from an elementary homological algebra argument involving the following diagram.

$$\begin{tikzcd}
H_1(P_{j+2}-C;\Lambda) \arrow[r] \arrow[d] & H_1(M-C;\Lambda) \arrow[r, "f_{j+2}"] \arrow[d, equal] & H_1(M-C,P_{j+2}-C;\Lambda) \arrow[r] \arrow[d, "c_{j+1}"] & \widetilde{H}_0(P_{j+2}-C)=0 \arrow[d] \\
H_1(P_{j+1}-C;\Lambda) \arrow[r, "e_{j+1}"] \arrow[d, "b_j"'] & H_1(M-C;\Lambda) \arrow[r, "f_{j+1}"] \arrow[d, equal] & H_1(M-C,P_{j+1}-C;\Lambda) \arrow[r] \arrow[d] & \widetilde{H}_0(P_{j+1}-C)=0 \arrow[d] \\
H_1(P_j-C;\Lambda) \arrow[r, "e_j"] & H_1(M-C;\Lambda) \arrow[r] & H_1(M-C,P_j-C;\Lambda) \arrow[r] & \widetilde{H}_0(P_j-C)=0
\end{tikzcd}$$

For the sake of completeness we include the details of the argument. Let f_{j+2}, f_{j+1}, e_{j+1}, and e_j be the indicated homomorphisms in the diagram. Choose $x \in H_1(M-C;\Lambda)$. We must show that there exists $u \in H_1(M-C;\Lambda)$ such that $x = (t-1)u$. By Lemma 5.2, there exists $y \in H_1(M - C, P_{j+1} - C; \Lambda)$ such that $c_{j+1}f_{j+2}(x) = (t - 1)y$. Hence $f_{j+1}(x) = (t - 1)y$. By exactness f_{j+1} is onto, so there exists $z \in H_1(M - C; \Lambda)$ such that $f_{j+1}(z) = y$. Now

$$f_{j+1}\big(x - (t - 1)z\big) = f_{j+1}(x) - (t - 1)f_{j+1}(z) = 0.$$

Thus there exists $w \in H_1(P_{j+1} - C; \Lambda)$ such that $e_{j+1}(w) = x - (t - 1)z$. Finally, another application of Lemma 5.2 gives $v \in H_1(P_j - C; \Lambda)$ such that $b_j(w) = (t - 1)v$. It is simple to check that $u = z + e_j(v)$ satisfies $x = (t - 1)u$.

This completes the proof of Theorem 1.1 in case $n = 4$.

7. Proof of Theorem 1.1 in case $n \geqslant 6$

In this section we prove Theorem 1.1 in the high dimensional cases. The proof follows the same outline as the 4-dimensional proof does. Rather than repeat all the details, we will go through the outline of the proof and indicate which parts are the same as in the previous proof and which details need to be changed.

The manifold W^n is simply $W^n = W \times S^{n-4}$, where W is the 4-dimensional example. Since $W \subset \mathbb{R}^4$, we have $W^n \subset \mathbb{R}^4 \times S^{n-4}$. But $\mathbb{R}^4 \times S^{n-4}$ embeds in S^n (as a regular neighborhood of $S^{n-4} \subset S^n$), so we have $W^n \subset S^n$. Suppose there exists a compact set $X^n \subset W^n$ such that $X^n \hookrightarrow W^n$ is a Čech equivalence. By compactness of X^n there must exist an i such that $X^n \subset \operatorname{Int} W(L_i) \times S^{n-4}$. This i will be fixed for the remainder of the proof. Of course, $W(L_i) \times S^{n-4} \subset M(L_i) \times S^{n-4}$. Let us use M^n to denote $M(L_i) \times S^{n-4}$. We have

$$\partial M = \partial\big(M(L_i) \times S^{n-4}\big) = \big(\partial M(L_i)\big) \times S^{n-4},$$

so we may define

$$\gamma : \pi_1(\partial M) \to \mathbb{J}$$

to be the composition of the homomorphism $\pi_1(\partial M) \to \pi_1(\partial M(L_i))$ induced by the project map with the homomorphism $\pi_1(\partial M(L_i)) \to \mathbb{J}$ that was use in the proof of the 4-dimensional case. One key point to notice is that the infinite cyclic covers satisfy

$$\widetilde{\partial M} = \big(\widetilde{\partial M(L_i)}\big) \times S^{n-4}.$$

The reason for this is the fact that $n \geqslant 6$ and so S^{n-4} is simply connected. The computation in [14] shows that $H_1(\partial M; \Lambda)_{(t-1)} \neq \{0\}$. We will use the existence of X^n to show that $H_1(\partial M; \Lambda)_{(t-1)} = \{0\}$ and thus derive our contradiction.

Using the fact that $X^n \hookrightarrow W^n$ is a Čech equivalence we can construct a sequence $N_1^n, N_2^n, N_3^n, \ldots$ of connected neighborhoods of X^n that are similar to those used in the proof of the 4-dimensional case. In particular, the neighborhoods will satisfy the following conditions.

(7.1) $N_{j+1}^n \subset \operatorname{Int} N_j^n$ for each $j \geqslant 1$.

(7.2) $\bigcap_{j=1}^{\infty} N_j^n = X^n$.

(7.3) The inclusion induced homomorphism $H_k(N_{j+1}^n) \to H_k(N_j^n)$ is zero for $k \neq 0, 2, n-4$, or $n-2$.

(7.4) If $\alpha_j^k : H_k(N_{j+1}^n) \to H_k(N_j^n)$ denotes the inclusion induced homomorphism, then $\operatorname{im} \alpha_j^k \cong \mathbb{Z}$ for $k = 0, 2, n-4$, or $n-2$. (Unless $n = 6$, in which case $\operatorname{im} \alpha_j^2 \cong \mathbb{Z} \oplus \mathbb{Z}$.)

(7.5) If $\beta_j^k : H_k(N_j^n) \to H_k(W^n)$ denotes the inclusion induced homomorphism, then $\beta_j^k | \operatorname{im} \alpha_j^k : \operatorname{im} \alpha_j^k \to H_k(W^n)$ is an isomorphism for all k.

As in Section 3, these properties imply the following lemma. The proof of Lemma 7.1 is the same as that of Lemma 3.1, except that (7.4) and (7.5) for the case $k = n-2$

must be used in place of (3.4) and (3.5). Let b_i denote the boundary of the cocore of the 2-handle in $W(L_i)$, just as before.

Lemma 7.1. *For each j, the inclusion induced homomorphism $\alpha'_j : H_1(N^n_{j+1} - X^n) \to H_1(N^n_j - X^n)$ satisfies $\operatorname{im}\alpha'_j \cong \mathbb{Z}$ and the inclusion induced homomorphism $\beta'_j : H_1(N^n_j - X^n) \to H_1(W^n - X^n)$ restricts to an isomorphism*

$$\beta'_j | \operatorname{im}\alpha'_j : \operatorname{im}\alpha'_j \xrightarrow{\cong} H_1(W^n - X^n).$$

Moreover, a generator of $H_1(W^n - X^n)$ is represented by $b_i \times \{\text{point}\}$.

Now $(M(L_i) \times S^{n-4}) - (W(L_i) \times S^{n-4}) \cong S^2 \times B^2 \times S^{n-4}$. Let us use Q to denote the core, $S^2 \times S^{n-4}$, of $(M(L_i) \times S^{n-4}) - (W(L_i) \times S^{n-4})$. As before, we connect X^n and Q with a PL ray A to form a new connected compactum $C^n = X^n \cup A \cup Q$. It is at this point in the proof that the major difference between this proof and the 4-dimensional one occurs. Note that the best we can hope for is that $\check{H}_*(C^n) \cong \check{H}_*((S^2 \times S^{n-4}) \vee (S^2 \times S^{n-4}))$ while $\check{H}_*(M^n) \cong \check{H}((S^2 \vee S^2) \times S^{n-4})$. Thus we cannot expect that $H_*(M^n - C^n, \partial M^n) = 0$. But we do not need the full strength of Lemmas 4.1 and 4.2; the weaker version stated as Lemma 7.2, below, suffices. Before stating the lemma, we must examine the neighborhoods of C^n.

Just as in Section 4, we construct a sequence of neighborhoods of C^n. For each j, let P^n_j denote a neighborhood of C^n obtained by taking the union of N^n_j and a regular neighborhood of $(A - \operatorname{Int} N^n_j) \cup Q$. The neighborhoods can be constructed to have the following properties.

(7.6) $P^n_{j+1} \subset \operatorname{Int} P^n_j$ for each $j \geqslant 1$.

(7.7) $\bigcap_{j=1}^{\infty} P^n_j = C^n$.

(7.8) The inclusion induced homomorphism $H_k(P^n_{j+1}) \to H_k(P^n_j)$ is zero for $k \neq 0, 2, n-4$, or $n-2$.

(7.9) If $\phi^k_j : H_k(P^n_{j+1}) \to H_k(P^n_j)$ denotes the inclusion induced homomorphism, then $\operatorname{im}\phi^k_j \cong \mathbb{Z} \oplus \mathbb{Z}$ for $k = 0, 2, n-4$ or $n-2$. (Unless $n = 6$, in which case $\operatorname{im}\phi^2_j \cong \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z}$, or $k = 0$, in which case $\operatorname{im}\phi^0_j \cong \mathbb{Z}$.)

(7.10) If $\psi^k_j : H_k(P^n_j) \to H_k(M^n)$ denotes the inclusion induced homomorphism, then $\psi^k_j | \operatorname{im}\phi^k_j : \operatorname{im}\phi^k_j \to H_k(M^n)$ is an isomorphism for $k = 0, 2$ or $n-2$ and is an epimorphism with kernel $\mathbb{Z}$ when $k = n-4$.

Lemma 7.2. $H_k(M^n - C^n, \partial M^n) = 0$ *for* $k \leqslant 2$.

Proof. Alexander duality (used as in the proof of Lemma 4.2) and the definition of Čech cohomology give

$$H_k(M^n - C^n, \partial M^n) \cong \check{H}^{n-k}(M^n, C^n) = \lim_{j \to \infty} H^{n-k}(M^n, P^n_j).$$

By the Universal Coefficient Theorem for Cohomology, it suffices to prove that $H_{n-k}(M^n, P^n_{j+1}) \to H_{n-k}(M^n, P^n_j)$ is the zero homomorphism for $k \leqslant 2$ and $\operatorname{im}[H_{n-3}(M^n, P^n_{j+1}) \to H_{n-3}(M^n, P^n_j)]$ is free abelian.

The fact that $H_{n-1}(M^n, P^n_{j+1}) \to H_{n-1}(M^n, P^n_j)$ is zero follows from the same argument as was used in the proof of Lemma 4.1 to show that $H_3(M, P_{j+1}) \to H_3(M, P_j)$ is zero. The fact that $H_{n-2}(M^n, P^n_{j+1}) \to H_{n-2}(M^n, P^n_j)$ zero follows from the same argument as was used in the proof of Lemma 4.1 to show that $H_2(M, P_{j+1}) \to H_2(M, P_j)$ is zero. The proof that $\operatorname{im}[H_{n-3}(M^n, P^n_{j+1}) \to H_{n-3}(M^n, P^n_j)]$ is free abelian follows from Property 7.9, above, along with the following diagram.

$$\begin{tikzcd}
0 = H_{n-3}(M^n) \arrow[r] \arrow[d, equal] & H_{n-3}(M^n, P^n_{j+1}) \arrow[r, "\delta^{n-3}_{j+1}"] \arrow[d, "\theta_j"'] & H_{n-4}(P^n_{j+1}) \arrow[r, "\psi^{n-4}_{j+1}"] \arrow[d, "\phi^{n-4}_j"] & H_{n-4}(M^n) \arrow[d, equal] \\
0 = H_{n-3}(M^n) \arrow[r] & H_{n-3}(M^n, P^n_j) \arrow[r, "\delta^{n-3}_j"] & H_{n-4}(P^n_j) \arrow[r, "\psi^{n-4}_j"] & H_{n-4}(M^n)
\end{tikzcd}$$

Let δ^{n-3}_{j+1}, δ^{n-3}_j and θ_j denote the homomorphisms indicated in the diagram. By Property 7.10, $\ker[\psi^{n-4}_j] \cap \operatorname{im}[\phi^{n-4}_j] \cong \mathbb{Z}$. Now

$$\operatorname{im}[\delta^{n-3}_j \theta_j] \subset \operatorname{im}[\phi^{n-4}] \cap \ker[\psi^{n-4}_j] \cong \mathbb{Z},$$

and δ^{n-3}_j is monic (by exactness), so $\operatorname{im}[\theta_j]$ is isomorphic to a subgroup of $\mathbb{Z}$ and hence is either $\{0\}$ or isomorphic to $\mathbb{Z}$. □

We can now complete the proof of Theorem 1.1. Lemma 7.2 allows us to extend γ to $M^n - C^n$ and thus we have an infinite cyclic cover of $M^n - C^n$. Exactly as in the 4-dimensional case, Lemma 7.2 together with [18, Lemma 3.5] imply that

$$H_1(\partial M^n; \Lambda)_{(t-1)} \cong H_1(M^n - C^n; \Lambda)_{(t-1)}.$$

The proof will be complete when we show that $H_1(M^n - C^n; \Lambda)_{(t-1)} = \{0\}$.

The fact that $H_1(M^n - C^n; \Lambda)_{(t-1)} = \{0\}$ is proved exactly as in the 4-dimensional case. The homological algebra argument for this was given in Section 6. That argument required Lemmas 5.1 and 5.2 as input. The proofs of those two lemmas, given is Section 5, required Lemma 3.1 as input. Now Lemma 7.1 is an n-dimensional version of Lemma 3.1, so Lemma 7.1 implies n-dimensional versions of Lemmas 5.1 and 5.2. Those lemmas in turn make the argument of Section 6 work and so the proof is complete. □

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