5-15-2002

Representing homology classes of simply connected 4-manifolds

Vo Thanh Liem
The University of Alabama

Gerard A. Venema
Calvin University

ELSEVIER

Topology and its Applications 120 (2002) 57–65

TOPOLOGY AND ITS APPLICATIONS

www.elsevier.com/locate/topol

Representing homology classes of simply connected 4-manifolds

Vo Thanh Liem [a], Gerard A. Venema [b,*,1]

[a] *Department of Mathematics, University of Alabama, Tuscaloosa, AL 35487-0350, USA*
[b] *Department of Mathematics and Statistics, Calvin College, Grand Rapids, MI 49546, USA*

Received 23 August 1999

To Benny Rushing

Abstract

The main theorem asserts that every 2-dimensional homology class of a compact simply connected PL 4-manifold can be represented by a codimension-0 submanifold consisting of a contractible manifold with a single 2-handle attached. One consequence of the theorem is the fact that every map of S^2 into a simply connected, compact PL 4-manifold is homotopic to an embedding if and only if the same is true for every homotopy equivalence. The theorem is also the main ingredient in the proof of the following result: If W is a compact, simply connected, PL submanifold of S^4, then each element of $H_2(W;\mathbb{Z})$ can be represented by a locally flat topological embedding of S^2.

AMS classification: 57N13; 57Q35

Keywords: Embedding; Homology class; 4-manifold

1. Introduction

Let W be a simply connected, compact, piecewise linear (PL) 4-dimensional manifold. (We allow manifolds to have boundary.) Each element of $H_2(W;\mathbb{Z})$ can be represented by an immersed PL 2-sphere. In this paper we study the problem of finding a better representative. In particular, we study the problem of finding a PL or topological embedding of S^2 that represents the specified homology class.

* Corresponding author. Current address: Division of Mathematical Sciences, National Science Foundation, 4201 Wilson Blvd., Room 1025, Arlington, VA 22230, USA.
E-mail addresses: vliem@ua1vm.ua.edu (V.T. Liem), venema@calvin.edu (G.A. Venema).
[1] The second author was partially supported by NSF grant number DMS-9626221.

0166-8641/02/$ – see front matter
PII: S0166-8641(01)00007-4

Problem 1.1. If $f : S^2 \to W^4$ is a continuous map of the 2-sphere into a simply connected, compact, PL 4-manifold, then is f homotopic to a PL (or topological) embedding?

Surprisingly, the answer to Problem 1.1 is not known in general for either PL or topological embeddings. It is well known that the answer is in general negative for both locally flat and smooth embeddings.

Consider a PL embedded 2-sphere. Such a 2-sphere may have a finite number of vertices at which it fails to be locally flat. If we run a PL arc (on the 2-sphere) through those vertices and then shrink the arc to a point, the result is a new PL 2-sphere with at most one non-locally flat point. A regular neighborhood of the 2-sphere consists of a 4-ball neighborhood of this distinguished vertex plus a single 2-handle attached along a knot in the boundary of the 4-ball. Conversely, any 4-manifold that is made up of a PL 4-ball with a single 2-handle attached contains a naturally embedded PL 2-sphere consisting of the core of the 2-handle together with the cone (in the 4-ball) on the attaching curve of the 2-handle. Thus a given homology class can be represented by a PL embedded 2-sphere if and only if it can be represented by a codimension-0 submanifold made up of a PL 4-ball (a 0-handle) with a single 2-handle attached. Our first theorem states that we can always achieve this if we use a homotopy 4-ball in place of B^4.

Theorem 1.2. *If W is a compact, simply connected, PL 4-manifold, then each element of $H_2(W;\mathbb{Z})$ can be represented by a compact 4-dimensional PL submanifold $M \subset W$ such that M consists of a Mazur-like contractible 4-manifold with a single 2-handle attached.*

Definition. A compact contractible PL 4-manifold is *Mazur-like* if it has a handle decomposition in which there is one 0-handle, no handles of index greater than 2, and the attaching map for the ith 2-handle is homotopic in the union of the 0- and 1-handles to the loop represented by the ith 1-handle.

These manifolds are called Mazur-like because they are very much like the famous contractible manifold of Mazur [9]. Fig. 1 shows a handle diagram of a typical example.

The manifold M represents a specified element of $H_2(W;\mathbb{Z})$ in the sense that a generator of $H_2(M;\mathbb{Z}) \cong \mathbb{Z}$ is homologous in W to the given element of $H_2(W;\mathbb{Z})$.

The theorem shows that the general case of Problem 1.1 is equivalent to the special case in which W has the homotopy type of S^2. The following problem was first raised by Matusmoto in the 1970s and is stated as Problem 4.25 in [6].

Problem 1.3 (*Matsumoto*). If $f : S^2 \to M$ is a homotopy equvalence from the 2-sphere to a compact PL 4-dimensional manifold, then is f homotopic to a PL (or topological) embedding?

Corollary 1.4. *Problems* 1.1 *and* 1.3 *are equivalent.*

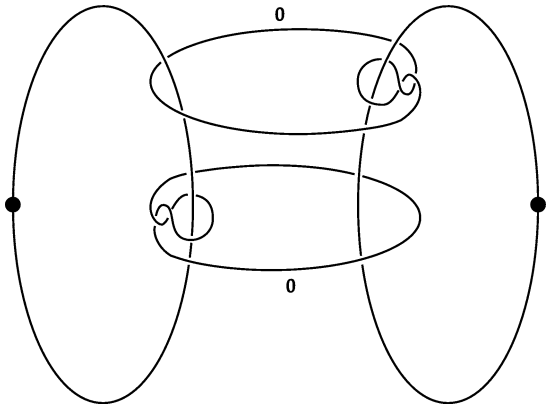

Fig. 1.

One case in which we can find an embedded 2-sphere is that in which the 4-manifold W is a subset of the 4-sphere. Theorem 1.2 is the main ingredient in the proof of the following result, which gives a positive solution to a special case of Matsumoto's problem.

Theorem 1.5. *If W is a compact simply connected PL submanifold of S^4, then each element of $H_2(W;\mathbb{Z})$ can be represented by a locally flat topological embedding of S^2.*

Obviously this theorem would not hold without the hypothesis that W embeds in S^4. For example, there are elements of $H_2(S^2 \times S^2)$ that cannot be represented by locally flat 2-spheres [11]. At the same time, every element of $H_2(S^2 \times S^2)$ can be represented by a PL embedded 2-sphere [7]. In addition, Akbulut [1] has shown that there is a compact PL 4-manifold W such that W has the homotopy type of the connected sum of a homology 4-ball and a copy of the complex projective plane but the generator of $H_2(W;\mathbb{Z})$ cannot be represented by a PL embedded 2-sphere (not even one with non-locally flat points).

Remark. It should be noted that both Theorems 1.2 and 1.5 are false without the hypothesis that W is compact. Specifically, there exists an open subset W of S^4 such that W has the homotopy type of S^2, but there is no compact subset $X \subset W$ such that $X \hookrightarrow W$ is a homotopy equivalence. In particular, there is no topologically embedded 2-sphere that represents a generator of $H_2(W;\mathbb{Z})$ nor is there a compact submanifold of the sort described in Theorem 1.2. The manifold W is constructed in [10] and the fact that it has the properties specified is proved in [12] and [8].

Before proceeding to the proof of Theorem 1.2, we digress to consider the relationship between Problem 1.1 and a related problem regarding contractible 4-manifolds.

Problem 1.6 (*Zeeman*). If V is a compact, contractible 4-manifold, then must every loop on the boundary of V bound a topologically embedded disk in V?

Problem 1.6 is exactly analogous to Problem 1.3 since it can be reformulated to ask whether or not every homotopy equivalence from B^2 to a compact 4-manifold which embeds boundary in boundary is homotopic, rel boundary, to an embedding. This question was first raised by Zeeman [13] in his famous paper on the Mazur contractible 4-manifold. Although Zeeman does not say so explicitly, his conjecture [13, Conjecture (5)] is usually interpreted to mean that there are PL loops on the boundary of the Mazur manifold that do not bound PL embedded disks in the manifold. In this interpretation the conjecture was solved by Akbulut [2]. Akbulut shows that there are loops on the boundary of the Mazur manifold (in fact the very ones identified by Zeeman) that do not bound PL disks in the Mazur manifold, not even PL disks with non-locally flat points. But it is clear in Akbulut's proof that these particular loops do bound topological disks in the Mazur manifold. Thus the topological version of Zeeman's problem remains open.

A final corollary of Theorem 1.2 is the fact that Zeeman's problem implies Matsomoto's problem.

Corollary 1.7. *A positive solution to Problem* 1.6 *implies a positive solution to Problem* 1.3.

The technical arguments in the proofs of Theorems 1.2 and 1.5 use delicate adjustments to a handle decomposition of W that are similar to those in [3].

2. Constructing the Mazur-like submanifold

In this section we prove Theorem 1.2. Without loss of generalitiy we may assume that $\partial W \neq \emptyset$.

Let α be a specified element of $H_2(W;\mathbb{Z})$. Since W is simply connected, the Hurewicz Theorem guarantees that α can be represented by a map of S^2 into $\operatorname{Int} W$. Make the map PL and throw it into general position. Let Σ denote the image of such a map. Then Σ is homeomorphic to the space obtained by identifying a finite number of pairs of points on S^2. Let Y_0 denote a close regular neighborhood of Σ. Form a handle decomposition $\mathcal{H}_0$ of Y_0 such that $\mathcal{H}_0$ has one 0-handle, a finite number of 1-handles, one 2-handle, and no handles of index greater than 2. We use N_0 to denote the union of the 0-handle and all the 1-handles. Note that the single 2-handle is attached along a homotopically trivial curve.

Now let $\mathcal{H}_1$ denote a handle decomposition of W that extends $\mathcal{H}_0$. By making use of the usual handle cancellation techniques, we may arrange that $\mathcal{H}_1$ contains only one 0-handle (the one in $\mathcal{H}_0$) and has no handles of index 4. Let Y_1 denote the union of Y_0 and all the 1-handles in $\mathcal{H}_1$ and let N_1 denote the union of N_0 and all the 1-handles in $\mathcal{H}_1$. Note that Y_1 has the homotopy type of $S^2 \vee (S^1 \vee S^1 \vee \cdots \vee S^1)$. In particular, $\pi_1(Y_1) \cong \pi_1(N_1)$ is a finitely generated free group with one generator for each 1-handle in $\mathcal{H}_1$. To be specific, the generator corresponding to the ith 1-handle is represented by a loop that goes exactly once over that 1-handle and misses all the other 1-handles. Since $Y_1 \hookrightarrow W$ induces the zero map on the fundamental group, it must be the case that the boundaries of the 2-handles

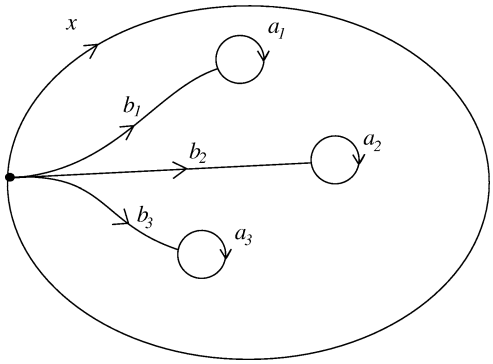

Fig. 2.

normally generate $\pi_1(Y_1)$. Thus each generator of $\pi_1(Y_1)$ is homotopically killed by some combination of 2-handles.

In order to make that last statement precise, we consider a specified generator x of $\pi_1(Y_1)$. Then x is null-homotopic in W, so x bounds a singular disk in W. This disk may be pushed off the 3-handles and put in general position with respect to the cocores of the 2-handles. The disk will then intersect the cocores in a finite number of points. We isotope the disk so that it has a finite number of disjoint subdisks in its interior, one for each of the points of intersection with the cocores, such that each subdisk is parallel to the core of one of the 2-handles and the remainder of the disk is contained in N_1. Note that the 2-handle in $\mathcal{H}_0$ need not be used because its boundary is null-homotopic in N_0. Let $a_1, a_2, \ldots, a_k$ denote the boundaries of the subdisks and let b_i denote a path on the disk from a basepoint in the disk to a_i. We may assume that two different b_i have only the basepoint in common. Then x is homotopic in N_1 to the loop $a_1^{b_1}a_2^{b_2}\cdots a_k^{b_k}$. Here a^b denotes the loop bab^{-1}. See Fig. 2.

Add a new cancelling $(2,3)$-handle pair to $\mathcal{H}_1$ near the base point. For each i, $1 \leqslant i \leqslant k$, slide the new 2-handle along b_i and over the 2-handle attached to a_i. The result is that the new 2-handle is now attached along the loop $a_1^{b_1}a_2^{b_2}\cdots a_k^{b_k}$. We construct one such new 2-handle for each of the finitely many generators of $\pi_1(N_1)$.

Now define V to be the union of N_1 and all the new 2-handles (one for each generator of $\pi_1(N_1)$). Note that V is contractible because N_1 has a 1-dimensional spine and the attaching curve for the jth 2-handle is homotopic in N_1 to the jth generator of $\pi_1(N_1)$. Define M to be the union of V and the original 2-handle in $\mathcal{H}_0$. Then M satisfies the conclusion of Theorem 1.2 and the proof is complete.

3. π_1-negligible submanifolds of S^4

In this section we prove Theorem 1.5. We assume for the entire section that W is a 4-manifold satisfying the hypotheses of Theorem 1.5. In particular, W is a compact codimension-0 submanifold of S^4.

Definition. A subset X of S^4 is said to be π_1-*negligible* in S^4 if $\pi_1(S^4 - X) = 0$.

Lemma 3.1. *If W is a subset of S^4, then the contractible submanifold V in the proof of Theorem* 1.2 *can be constructed so that V is π_1-negligible in S^4.*

Proof of Theorem 1.5 (assuming Lemma 3.1). Let M and V be as in the proof in the previous section. By Lemma 3.1 we may assume that V is π_1-negligible. Since V is contractible, the long exact sequence of the pair $(V, \partial V)$ shows that ∂V is a homology 3-sphere and a similar sequence shows that $U = S^4 - \operatorname{Int} V$ is a homology 4-ball. But the fact that V is π_1-negligible means that U is simply connected. Hence U is contractible. By [5, Theorem 1.4′], ∂V bounds a unique contractible 4-manifold. (See also Proposition 11.6A or the last sentence on p. 204 of [4].) Thus there is a (topological) homeomorphism $h : U \to V$ such that $h|\partial V = \mathrm{id}$. Let D be the core of the 2-handle $M - \operatorname{Int} V$. Then $\Sigma = D \cup h(D)$ is the locally flat topological 2-sphere we need to satisfy the conclusion of Theorem 1.5. □

Definition. Let V be a compact PL codimension-0 submanifold of S^4 and let $h^{(2)}$ be a 2-handle in some handle decomposition of V. An *immersed transverse sphere for $h^{(2)}$* is an immersed PL 2-sphere $\Sigma \subset S^4$ such that $\Sigma \cap V$ equals the cocore of $h^{(2)}$.

The proof of Lemma 3.1 is based on a simple observation.

Observation 3.2. *Let V be a compact PL codimension-*0 *submanifold of S^4 such that V has a handle decomposition containing no handles of index greater than* 2. *If $V' \subset S^4$ is obtained from V by attaching* 1- *and* 2*-handles and V' is π_1-negligible in S^4, then V is π_1-negligible in S^4.*

Proof. Let V be a compact codimension-0 submanifold of S^4 such that V has a handle decomposition containing no handles of index greater than 2. It is easy to see that such a V is π_1-negligible if and only if every 2-handle in V has an immersed transverse sphere. Take a handle decomposition of V and extend it to a decomposition of V'. Any 2-handle in V is also a 2-handle in V' and therefore has an immersed sphere that is transverse to V'. The same immersed sphere is also an immersed transverse sphere for the 2-handle thought of as a 2-handle in V. □

The strategy of the proof of Lemma 3.1 is to prove that the submanifold V constructed in the previous section can be enlarged to a π_1-negligible manifold V' with a 2-dimensional spine. Observation 3.2 then shows that V is π_1-negligible. Note that it is not necessary to have $V' \subset W$. We need only prove that V is π_1-negligible in S^4, so $V' \subset S^4$ will suffice. For the remainder of the proof we work in the ambient manifold S^4 and do not concern ourselves with W any more after the initial setup of the proof.

Proof of Lemma 3.1. Let $W \subset S^4$ be as in the hypothesis of Theorem 1.2. This proof builds on the proof of that theorem, so we use the same notation. In particular, Σ, N_0, N_1,

Y_0, Y_1, $\mathcal{H}_0$, and $\mathcal{H}_1$ all denote the same things as in that proof. Extend $\mathcal{H}_1$ to a handle decomposition $\mathcal{H}$ of S^4. Cancel all but one 0-handle and all but one 4-handle. Define N to be the union of the 0-handle and all the 1-handles; define N^* to be the union of the 4-handle and all the 3-handles in $\mathcal{H}$. Note that both N and N^* collapse to 1-dimensional spines. Hence $\pi_1(N)$ and $\pi_1(N^*)$ are finitely generated free groups. Furthermore, the inclusion-induced homomorphism $\pi_1(N) \to \pi_1(S^4 - \operatorname{Int} N^*)$ is zero. Define Y to be $N \cup H$ where H is the distinguished 2-handle (that carries the homology class represented by Σ). Since H is attached along a homotopically trivial curve in N, the inclusion induced homomorphism $\pi_1(N) \to \pi_1(Y)$ is an isomorphism.

Let n denote the free rank of $\pi_1(N)$. Add n cancelling $(2,3)$-handle pairs to the handle decomposition $\mathcal{H}$. Then slide the new 2-handles into position so that the ith 2-handle homotopically cancels the ith original 1-handle. (This operation is exactly the same as that described in the proof of Theorem 1.2, except that there are more handles.) Define V_1 to be the union of N and these new 2-handles. Note that V_1 is contractible. Further, the process used to create V_1 is exactly the same as that used in the proof of Theorem 1.2 to create V, so we can certainly construct V_1 so that $V \subset V_1$ and all the handles of V are handles of V_1. Thus, by Observation 3.2, the proof will be complete if we can show that V_1 is π_1-negligible.

Let $H_1, H_2, \ldots, H_{n+k}$ denote the 2-handles of $\mathcal{H}$. Each H_i determines a pair $(g_i, h_i) \in \pi_1(N^*) \times \pi_1(N)$, where g_i is the boundary of the cocore of H_i and h_i is the boundary of the core. The handle slides in the previous paragraph made $h_1, h_2, \ldots, h_n$ the standard generators of $\pi_1(N)$. By further 2-handle slides, we may arrange that $h_i = 1$ for $i > n$. Let us say that $H_{n+1} = H$, the distinguished 2-handle that is part of Y. Note that it is already the case that $h_{i+1} = 1$, so there is no need to modify H in the last step above.

Define $F = \pi_1(N^*)$, which is a free group of rank $r \leqslant k$, and define $C = [F, F]$, the commutator subgroup. Now $F/C \cong H_1(N^*)$, is a free abelian group of rank r. In addition, $H_1(S^4 - \operatorname{Int} V_1) = 0$ (because V_1 and $S^4 - \operatorname{Int} V_1$ are homology balls), so $g_{n+1}, g_{n+2}, \ldots, g_{n+k}$ generate F/C. Let us say that the standard generators are $x_1, \ldots x_r$. We may assume that g_{n+1} is one of the standard generators of F. If this is not the case, add a new $(2,3)$-handle pair such that the belt sphere of the 3-handle goes around g_{n+1} and the 2-handle represents the commutator of g_{n+1} with the new generator of F. Think, dually, of H as a 2-handle H^* attached to N^* and slide it over the new 2-handle just introduced. The result is that g_{n+1} equals the new generator of F. Furthermore, these operations can be performed in a small neighborhood of H and do not change Σ.

Now each of the standard generators of F/C can be expressed as a sum of g_{n+1}, $g_{n+2}, \ldots, g_{n+k}$. Let us say that $x_1 = g_{n+1}$. Do handle slides to change g_i, $i \geqslant n+2$, so that $g_{n+2}, \ldots, g_{n+k}$ generate the orthogonal complement of x_1 in F/C. Express x_2 in terms of $g_{n+2}, g_{n+3}, \ldots, g_{n+k}$. By doing handle slides, we can reduce the absolute value of the largest coefficient until it is 1. Then further handle slides can be performed until x_2 is equal to one of the g_i. A sequence of handle slides of this type will make $g_{n+1}, g_{n+2}, \ldots, g_{n+r}$ the standard generators of F/C. These handle slides involve only handles of index greater than $n+1$, so V_1 is left unchanged. The end result is that there exist $c_1, \ldots, c_r \in C$ such that

$\{g_{n+1}c_1, g_{n+2}c_2, \ldots, g_{n+r}c_r\}$ are the standard generators of F and we have been careful enough so that $c_1 = 1$.

Stabilize by adding r cancelling $(1,2)$-handle pairs to $\mathcal{H}$. Enlarge N by adding all the new 1-handles. The new 2-handles then have curves $(1, e_i)$, where $h_1, \ldots, h_n, e_1, \ldots, e_r$ are the free generators of $\pi_1(N)$. At the same time we add all the new 1- and 2-handles to V_1 to form a larger Mazur-like contractible 4-manifold V_2. Applying Observation 3.2 again, we see that the proof of the lemma will be complete if we can show that V_2 is π_1-negligible in S^4.

Now consider the dual handle decomposition $\mathcal{H}^*$. The 2-handles of $\mathcal{H}$ are also 2-handles in $\mathcal{H}^*$, but with the cores and cocores interchanged. Do a sequence of the double handle slides of [3] to the $(1, e_i)$ handles to change the attaching curve for the ith one to (c_i, e_i). (This "double handle slide" is the sequence of two handle slides described in the first full paragraph on p. 346 of [3].) These moves may be viewed as handle slides of V_2. Each double handle slide consists of sliding one of the new 2-handles in $\mathcal{H}$ back and forth over some other handle. Dually in $\mathcal{H}^*$, we see some handle sliding back and forth over one of the new 2-handles. (Any 2-handle slide can be viewed as a slide in $\mathcal{H}$ or as a slide in $\mathcal{H}^*$. The relationship between the two views is explained on p. 345 of [3].) In particular, the moves can be accomplished with an ambient isotopy of V_2. Obviously V_2 is π_1-negligible if and only if some isotopic image of it is.

We now do one final set of handle slides. We slide the handle with curves $(g_{n+i}, 1)$ over the handle with curves (c_i, e_i) so that the pair $(g_{n+i}, 1)$ is replaced by $(g_{n+i}c_i, 1)$. Dually, this means that the handle representing (c_i, e_i) is slid over that representing $(g_{n+i}, 1)$. Since the latter handles are not part of V_2, this is not an isotopy of V_2. In fact, V_2 is replaced by a new manifold V_3. Note that V_3 is still a Mazur-like contractible manifold since the homotopy classes of the attaching curves have not changed. Furthermore, V is a submanifold of V_3 and V_3 is obtained from V by adding 1- and 2-handles. So it still suffices to prove that V_3 is π_1-negligible.

But the fact that V_3 is π_1-negligible is easy to see. It is so because $\pi_1(S^4 - V_3)$ is generated by $\pi_1(N^*)$, the curves $\{g_{n+i}c_i\}$ are the standard generators of $\pi_1(N^*)$, and $g_{n+i}c_i$ bounds one of the 2-handles of $S^4 - V_3$. □

References

[1] S. Akbulut, On representing homology classes of 4-manifolds, Invent. Math. 49 (1978) 193–198.

[2] S. Akbulut, A solution to a conjecture of Zeeman, Topology 30 (1991) 513–515.

[3] C.L. Curtis, M.H. Freedman, W.C. Hsiang, R. Stong, A decomposition theorem for h-cobordant smooth, simply-connected compact 4-manifolds, Invent. Math. 123 (1996) 343–348.

[4] M.H. Freedman, F. Quinn, Topology of 4-Dimensional Manifolds, Princeton Univ. Press, Princeton, NJ, 1990.

[5] M.H. Freedman, The topology of four-dimensional manifolds, J. Differential Geom. 17 (1982) 352–453.

[6] R.C. Kirby, Problems in low-dimensional topology, in: W.H. Kazez (Ed.), Geometric Topology, Part II, AMS/IP Studies in Advanced Mathemathics, Vol. 2, American Mathematical Society and International Press, Providence, RI, 1997, pp. 35–473.
[7] M.A. Kervaire, J.W. Milnor, On 2-spheres in 4-manifolds, Proc. Nat. Acad. Sci. USA 47 (1961) 1651–1657.
[8] V.T. Liem, Y. Matsumoto, G.A. Venema, A homotopy equivalence that is not homotopic to an embedding, Topology Appl. 90 (1998) 211–222.
[9] B. Mazur, A note on some contractible manifolds, Ann. of Math. 73 (1961) 221–228.
[10] Y. Matsumoto, G. Venema, Failure of the Dehn lemma on contractible 4-manifolds, Invent. Math. 51 (1979) 205–218.
[11] A.G. Tristan, Some cobordism invariants for links, Proc. Cambridge Philos. Soc. 66 (1969) 251–264.
[12] G.A. Venema, An example of manifold that does not contain a compact core, Topology Appl. 90 (1998) 197–210.
[13] E.C. Zeeman, On the dunce hat, Topology 2 (1964) 341–358.