Relations in the homotopy of simplicial abelian Hopf algebras

James M. Turner
Calvin University

Follow this and additional works at: https://digitalcommons.calvin.edu/calvin_facultypubs

Part of the Mathematics Commons

Recommended Citation
https://digitalcommons.calvin.edu/calvin_facultypubs/544

This Article is brought to you for free and open access by the University Faculty Scholarship at Calvin Digital Commons. It has been accepted for inclusion in University Faculty Publications by an authorized administrator of Calvin Digital Commons. For more information, please contact dbm9@calvin.edu.
Relations in the homotopy of simplicial abelian Hopf algebras

James M. Turner

Department of Mathematics and Statistics, Calvin College, 3201 Burton Street, S.E., Grand Rapids, MI 49546, USA

Communicated by J.D. Stasheff; received 13 September 1995; received in revised form 16 August 1994

Abstract

In this paper, we analyze the structure possessed by the homotopy groups of a simplicial abelian Hopf algebra over the field $\mathbb{F}_2$. Specifically, we review the higher-order structure that the homotopy groups of a simplicial commutative algebra and simplicial cocommutative coalgebra possess. We then demonstrate how these structures interact under the added conditions present in a Hopf algebra. © 1999 Elsevier Science B.V. All rights reserved.

1991 Math. Subj. Class.: 16W30; 18G30; 55S05

0. Introduction

The goal of this paper is to determine all the natural relations among primary operations that occur in the homotopy of a simplicial abelian Hopf algebra over $\mathbb{F}_2$, the field of two elements. Here abelian Hopf algebra means a unitary commutative algebra in the category of counitary cocommutative coalgebras.

The motivation for this problem comes studying the second quadrant cohomology spectral sequence, over a finite field, associated to a cosimplicial space (see [3]). The $E_2$-term of such a spectral sequence is the homotopy groups for the simplicial (unstable) algebra obtained from applying cohomology to this cosimplicial space. Given a simplicial commutative $\mathbb{F}_2$-algebra $A$, its homotopy groups possess operations

$$\delta_i : \pi_n A \rightarrow \pi_{n+i} A,$$

as constructed in [4, 2, 9]. As such they are called Cartan–Bousfield–Dwyer operations. In [10] it is shown that if $A$ arises from cohomology, as above, then, in the associated
spectral sequence, these operations guarantee that Steenrod operations, which would violate instability at $E_\infty$, do not survive.

Now if one started off with a cosimplicial iterated loop space, the associated cohomology is also a simplicial cocommutative coalgebra. The homotopy of such an object $B$ has a right action of Steenrod operations

$$Sq^i : \pi_n B \to \pi_n B$$

which can be extracted from [7]. If $B$ arises from cohomology, as above, then these Steenrod operations determine Dyer–Lashof operations in the abuttment. This is shown in [19, 20] examines a specific example.

Now the cohomology of a cosimplicial iterated loop space is more than just a simplicial algebra and a simplicial coalgebra, it is a simplicial Hopf algebra. Given such an object $H$ the added Hopf condition guarantees certain "Nishida relations" amongst the above operations, in the homotopy. The determination of these relations is the objective of this paper.

We organize this paper as follows. In Section 1, we review the primary structure possessed by the homotopy of simplicial commutative algebras and simplicial cocommutative coalgebras. This is codified through the notions of $D$-algebra and $A$-coalgebra respectively. We then define \textit{Hopf $D$-algebra} and state the main theorem that says this correctly describes the relations that occur in the homotopy of a simplicial abelian Hopf algebra. In Section 2, we describe a natural map between composite functors of vector spaces. This natural map is useful in describing a modified Hopf condition for abelian Hopf algebras. We also describe a simple relation that occurs in a Hopf $\Gamma$-algebra. In Section 3, we review the homotopy of the symmetric (co)-invariants on a simplicial vector space. We use this, in Section 4, to reduce the proof of the main theorem to showing certain relations occur in the homotopy of the symmetric invariants on any simplicial commutative algebra. This is reduced further to computing the effect, in homotopy, of the natural map of Section 2, prolonged to simplicial vector spaces. In Section 5, we show that such a calculation is determined by the effect, in homotopy, of a natural idempotent on the $D_8$-invariants acting on the fourfold tensor product of a simplicial vector space (here $D_8$ is the dihedral group of order 8, viewed as a subgroup of the symmetric group on four letters). In Section 6, we review some needed tools from group cohomology plus a useful theorem from [9] which allows us to prove our final reduction to a group cohomological calculation. This calculation is performed in Section 7.

1. Simplicial algebras, coalgebras, Hopf algebras, and their homotopy

In this section, we review the structure of the homotopy groups for simplicial commutative algebras and simplicial cocommutative coalgebras. This is expressed through the notions of $D$-algebras and $A$-coalgebras, respectively. As an immediate consequence the homotopy groups of a simplicial bicommutative Hopf algebra possesses both of
these structures which the Hopf condition tells us must interact in a meaningful way. We codify through the notion of a Hopf D-algebra. The main theorem of this paper is that this notion accurately describes this expanded structure.

We denote the category of commutative algebras (respectively commutative graded algebras) by $\mathcal{A}$ (respectively $\mathcal{A}_s$).

Given a graded algebra $A$, let $I_s(A) \subseteq A_s \geq 0$ denote the ideal of elements $x$ in $A$ such that $|x| \geq s$.

**Definition 1.1.** A $\Gamma$-algebra is a commutative graded algebra $A$ together with a map $\gamma_2: \Phi I_2 \to A$
such that

1. $I_1$ is exterior under the product of $A$,
2. for $x, y \in I_2$
   \[ \gamma_2(x \cdot y) = \gamma_2(x) + \gamma_2(y) + x \cdot y, \]
3. for $x, y \in A$ such that $x \cdot y \in I_2$
   \[ \gamma_2(x \cdot y) = \begin{cases} 0, & x, y \in I_1, \\ (x \cdot x) \cdot \gamma_2(y), & |x| = 0, \\ \gamma_2(x) \cdot (y \cdot y), & |y| = 0. \end{cases} \]

We now make the following, as given in [12].

**Definition 1.2.** A $D$-algebra $A$ is a $\Gamma$-algebra together with maps $\delta_i: A_n \to A_{n+i}$ for all $2 \leq i \leq n$ such that

1. $\delta_i$ is a homomorphism, for $i < n$, and $\delta_n = \gamma_2$,
2. for $x, y \in A$ such that $x \cdot y \in A_n$ then
   \[ \delta_i(x \cdot y) = \begin{cases} (x \cdot x) \cdot \delta_i(y), & |x| = 0, \\ \delta_i(x) \cdot (y \cdot y), & |y| = 0, \\ 0 & \text{otherwise}. \end{cases} \]
3. for $x \in A_n$ and $j < 2i$ then
   \[ \delta_j \delta_i x = \sum_{\frac{i-j+s-1}{i-s} \leq s \leq \frac{i-j+s-1}{i-j}} \left( \begin{array}{c} i-j+s-1 \\ i-s \end{array} \right) \delta_{j+i-s} \delta_s x. \]

A map of $D$-algebras is a map in $\mathcal{A}_s$ that commutes with the $\delta_i$. We denote the category of $D$-algebras by $\mathcal{AD}$.

We denote the category of simplicial commutative algebras by $s\mathcal{A}$. 
The following was proved in [9, 13].

**Theorem 1.3.** Let \( A \) be a simplicial commutative algebra. Then \( \pi_* A \) is naturally a \( D \)-algebra, i.e. we have a functor
\[
\pi_* : sA \to DA.
\]

**Remark.** The operations \( \delta_i \) in (1.2) were first discovered in [4]. Their properties were subsequently derived in [2, 9]. In the latter, they were called higher divided squares. We will also refer to them as Cartan–Bousfield–Dwyer operations.

Next, denote the category of cocommutative coalgebras (resp. cocommutative graded coalgebras) by \( C.A \) (resp. \( C.A_* \)).

Given a cocommutative graded coalgebra \( \Pi \) we define the coalgebra map
\[
\xi : \Pi \to \Phi \Pi
\]
called the Verschiebung, as the linear dual of the squaring map (we will give a more meaningful definition in the next section).

**Definition 1.5.** An \( A \)-coalgebra is a cocommutative graded coalgebra \( \Pi \) together with homomorphisms
\[
\text{Sq}^i : \Pi_n \to \Pi_{n-i}
\]
for \( i \geq 0 \) such that for \( x \in \Pi_n \) we have
1. \( x \text{Sq}^i = 0 \) for \( 2i > n \) and \( x \text{Sq}^{n/2} = \xi(x) \),
2. if \( \Delta x = \Sigma x' \otimes x'' \) then
\[
\Delta(x \text{Sq}^i) = \sum_{s+t=i} \sum (x' \text{Sq}^s) \otimes (x'' \text{Sq}^t),
\]
3. for \( j < 2i \) we have
\[
x \text{Sq}^i \text{Sq}^j = \sum_{2s \leq j} \binom{i-s-1}{j-2s} x \text{Sq}^{i+j-s} \text{Sq}^s.
\]

We define a map of \( A \)-coalgebras to be a map in \( C.A_* \) which commutes with the \( \text{Sq}^i \). Denote by \( K^* \) the category of \( A \)-coalgebras.

**Note.** \( \mathcal{K} \) clearly denotes the Steenrod algebra.

We denote the category of simplicial cocommutative coalgebras by \( sC.A \). A consequence of [7] (see also [12]) is the following

**Theorem 1.6.** Let \( \Pi \) be a simplicial cocommutative coalgebra. Then \( \pi_* \Pi \) is naturally an \( A \)-coalgebra. That is, we have a functor
\[
\pi_* : sC.A \to K^*.
\]
Recall, now, that a (graded) Hopf algebra (in the sense of [17]) is a (graded) module $H$ which is both a (graded) algebra and a (graded) coalgebra for which the two diagrams

\[
\begin{align*}
\xymatrix{ 
H \otimes H \ar[r]^{\Delta \otimes \Delta} & H \otimes H \otimes H \otimes H \\
& & & 1 \otimes 1 \otimes 1 \\
& & & \text{m} \otimes m \\
H \ar[u]^{m} \ar[r]_{\Delta} & H \otimes H \ar[u]_{\text{m} \otimes \text{m}}
\end{align*}
\]

(1.7)

and

\[
\begin{align*}
\xymatrix{ 
F_2 \ar[r]^{\eta} & H \\
& & & 1 \\
& & & \epsilon \\
& & F_2 \ar[u]_{1}
\end{align*}
\]

(1.8)

commute. A map of Hopf algebras is simply a map of algebras and a map of coalgebras. We further define a Hopf algebra to be abelian if it is commutative as an algebra and cocommutative as a coalgebra.

We denote by $\mathcal{H}_A$ (resp. $\mathcal{H}_A^*$) the category of abelian Hopf algebras (resp. abelian graded Hopf algebras).

**Definition 1.9.** A Hopf $\Gamma$-algebra is a pair $(H, \gamma_2)$ consisting of an abelian graded Hopf algebra $H$ together with a map

$\gamma_2: \Phi_{l_2} \rightarrow H$

satisfying 1–3 of Definition 1.1 along with the additional condition 4 for $x \in l_2$

$\Delta \gamma_2 \alpha = \gamma_2(\Delta x)$.

A map of Hopf $\Gamma$-algebras is just a map in $\mathcal{H}_\Lambda$, which is also a map of $\Gamma$-algebras.

**Definition 1.10.** A Hopf $D$-algebra is a Hopf $\Gamma$-algebra $H$ together with maps

$\delta_i: H_n \rightarrow H_{n+i}$

for all $2 \leq i \leq n$, satisfying conditions 1–3 of Definition 1.2, and with maps

$Sq^i: H_n \rightarrow H_{n-i}$

for all $i \geq 0$, satisfying conditions 1–3 of Definition 1.5, such that the following relations are satisfied for a fixed $x \in H_n$: 
1. for each $2 \leq i \leq n$
\[ \Delta \delta_i x = \delta_i \Delta x \]
and for any $y \in H$, $j \geq 0$
\[ (x \cdot y) Sq^j = \sum_{s+j} (x Sq^s) \cdot (y Sq^j), \]

2. for each $2 \leq j < n$ and $i \geq 0$
\[
\begin{align*}
(\delta_j x) Sq^j &= \begin{cases} 
\sum_s (i - j, j - 2i + 2s - 1) \delta_{j-i+s}(x Sq^s), & i > j, \\
\xi_j \xi x + \sum_{2s > j} \delta_s(x Sq^s), & i = j, \\
\sum_s (i - 2s, j - 2i + 2s - 1) \delta_{j-i+s}(x Sq^s), & i < j,
\end{cases} \\
(\delta_n x) Sq^i &= \begin{cases} 
0, & i > n, \\
\xi_n \xi x, & i = n, \\
\sum_{2s < i} (x Sq^s)(x Sq^{i-s}) + \sum_s (i - 2s, n - 2i + 2s - 1) \delta_{n-i+s}(x Sq^s), & i < n.
\end{cases}
\end{align*}
\]

A map of Hopf $D$-algebras is simply a map of $D$-algebras and a map of $A$-coalgebras. We denote the category of Hopf $D$-algebras by $\mathcal{H}D$.

We denote by $s\mathcal{H}A$ the category of simplicial abelian Hopf algebras.

We now come to the main theorem of this work, whose proof is postponed to Section 4.

**Theorem 1.11.** Let $H$ be a simplicial abelian Hopf algebra. Then $\pi_* H$ is naturally a Hopf $D$-algebra. That is we have a functor

\[ \pi_* : s\mathcal{H}A \rightarrow \mathcal{H}D. \]

We close this section by noting some consequences of this theorem and of the notion a Hopf $D$-algebra in general. These will be proved in [14].

Since objects of $\mathcal{A}D$ are augmented commutative $\mathbb{F}_2$-algebras, the indecomposables functor defines a functor

\[ Q : \mathcal{A}D \rightarrow \mathcal{U}D \]

where the target is the category of unstable $D$-modules. Let $B$ be the algebra of Cartan–Bousfield–Dwyer operations. Then $\mathcal{U}D$ is the category of unstable $B$-modules and we have a functor

\[ \mathbb{F}_2 \otimes_B (-) : \mathcal{U}D \rightarrow \mathcal{M}\mathbb{F}_2. \]
Let
\[ Q_D : \mathcal{A} \rightarrow m \mathbb{F}_2 \]
be the composite of these two functors. By restriction we have a functor
\[ Q_D : \mathcal{H} \rightarrow \mathcal{L} \]
where \( \mathcal{L} \) is the category of right \( \mathcal{A}_0 \)-modules, \( \mathcal{A}_0 = \mathbb{F}_2[Sq^0, Sq^1]/[(Sq^1)^2] \). This follows from 2 of Definition 1.10. Let \( \mathcal{H}_+ \) be the subcategory of connected objects in \( \mathcal{H} \) (i.e. isomorphic to \( \mathbb{F}_2 \) in degree 0) and \( \mathcal{L}_+ \) be the subcategory of connected objects in \( \mathcal{L} \) (i.e. 0 in degree 0).

**Theorem 1.12.** The functor \( Q_D : \mathcal{H} \rightarrow \mathcal{L} \) possesses a right adjoint
\[ A : \mathcal{L} \rightarrow \mathcal{H} \]
such that the restricted adjoint pair
\[ Q_D : \mathcal{H}_+ \leftrightarrow \mathcal{L}_+ ; A \]
is an equivalence of categories.

An interesting input to the proof of this theorem is

**Proposition 1.13.** If \( H \in \mathcal{H}_+ \) then \( H \) is a free \( D \)-algebra.

To make this useful for simplicial abelian Hopf algebras we prove

**Theorem 1.14.** For \( H \) \( \pi_0 \)-connected in \( \text{s}\mathcal{H} \mathcal{A} \) we have
\[ Q_D(\pi_*H) \cong H_*^Q(H) \]
where the right-hand side is the André-Quillen homology of \( H \) as a simplicial algebra (see [12]).

As an application, let \( A \) be a connected simplicial abelian group. Then \( \mathbb{F}_2[A] \) is a connected object of \( \text{s}\mathcal{H} \mathcal{A} \) so from these results
\[ H_*(A; \mathbb{F}_2) \cong \pi_*\mathbb{F}_2[A] \cong A(H_*^Q(\mathbb{F}_2[A])) \]
and we have a short exact sequence
\[ 0 \rightarrow \mathbb{Z}/2 \otimes \pi_q A \rightarrow H_*^Q(\mathbb{F}_2[A]) \rightarrow Tor(\mathbb{Z}/2, \pi_{q-1} A) \rightarrow 0. \]
This recovers H. Cartan’s main computation in [4].
2. Group actions and consequences of commutativity

Let $V$ be an $\mathbb{F}_2$-module and define

$$V^\otimes m = V \otimes \cdots \otimes V.$$ 

Then $\Sigma_m$, the symmetric group on $m$ letters, acts on $V^\otimes m$ by permutation. Thus, for any subgroup $G \leq \Sigma_m$, $V^\otimes m$ is a $G$-module. With this we define the $G$-symmetric invariant functor

$$S^G : m\mathbb{F}_2 \to m\mathbb{F}_2$$

by

$$S^G(V) = (V^\otimes m)^G$$

and the $G$-symmetric coinvariant functor

$$S_G : m\mathbb{F}_2 \to m\mathbb{F}_2$$

by

$$S_G(V) = (V^\otimes m)_G.$$ 

If $G = \Sigma_m$ then we denote (2.1) by $S^m$ and (2.2) by $S_m$.

Now, let $N \in \mathbb{F}_2[G]$ be defined by

$$N = \sum_{g \in G} g.$$ 

Then the action of $N$ on $V^\otimes m$ defines a map which factors

$$V^\otimes m \xrightarrow{\tau} \xrightarrow{\overline{N}} V^\otimes m \xrightarrow{i} S^G V$$

but since, for any $x \in V^\otimes m$, $\tau(gx) = \tau(x)$ for any $g \in G$ then we have a further factorization,

$$V^\otimes m \xrightarrow{\tau} S^G V \xrightarrow{\rho} S^G V \xrightarrow{N}$$

defining the norm map $N$.

Because of its importance later, we analyze the norm map $N$ in the case $G = \Sigma_2$. First, we define the diagonal map

$$d : \Phi V \to V^\otimes 2.$$
by \( d\bar{x} = x \otimes x \). This is not a homomorphism, nonetheless we have a commuting diagram

\[
\begin{array}{cccccc}
& S^2 V & \rightarrow & & & \\
\sigma & \downarrow i & & \downarrow \rho & & \\
\Phi V & \rightarrow & V^\otimes 2 & \rightarrow & S_2 V & \\
\downarrow d & & \downarrow i & & \downarrow \rho & \\
& S_2 V & \\
\end{array}
\]

\( \sigma \) is not a homomorphism, but \( i \) is one. From this we define the exterior square functor \( E_2 \) by \( E_2 V = \text{coker } i \). We now have the following commutative diagram:

\[
\begin{array}{cccccc}
0 & \rightarrow & \Phi V & \rightarrow & S_2 V & \rightarrow & \text{coker } N & \rightarrow & 0 \\
\downarrow & & \downarrow \sigma & & \downarrow \zeta & & \downarrow \pi & & \downarrow 0 \\
& \Phi V & \rightarrow & E_2 V & \rightarrow & S^2 V & \\
\end{array}
\]

from which we have that \( E_2 V = \text{im } N = \ker \pi \). Note that \( \pi \sigma \) is a linear isomorphism.

As an application of (2.7) we have

**Proposition 2.8.** For any \( \omega \in S^2 V \) there exists \( \alpha \in E_2 V \) and \( x \in V \), uniquely determined by \( \omega \), such that

\[ \omega = v(\alpha) + \sigma(\bar{x}) \]

**Proof.** Let \( \tilde{\sigma} : \text{coker } N \rightarrow S^2 V \) be the composite \( \sigma \cdot (\pi \sigma)^{-1} \) so that \( \pi \tilde{\sigma} = 1 \). Then the self-map

\[ 1 + \tilde{\sigma} \pi : S^2 V \rightarrow S^2 V \]

satisfies \( \pi(1 + \tilde{\sigma} \pi) = 0 \). Thus since \( v \) is injective, there exists \( \alpha \in E_2 V \) such that

\[ v(\alpha) = \omega + \tilde{\sigma} \pi(\omega) \]

Finally, let \( x \in V \) be the element which satisfies

\[ \bar{x} = (\pi \sigma)^{-1}(\pi \omega) \]

The conclusion follows. \( \square \)

Now, given a commutative algebra \( A \) the product map

\[ m : A \otimes A \rightarrow A, \]
factors as

\[
\begin{array}{ccc}
\Lambda \otimes \Lambda & \xrightarrow{m} & \Lambda \\
\rho \downarrow & & \downarrow \\
S_2 \Lambda \\
\end{array}
\]  

(2.10)

Similarly, given a cocommutative coalgebra \( \Pi \) the coproduct map

\[
\Delta : \Pi \to \Pi \otimes \Pi,
\]

(2.11)

factors as

\[
\begin{array}{ccc}
\Pi & \xrightarrow{\Delta} & \Pi \otimes \Pi \\
\psi \downarrow & & \downarrow \\
S^2 \Pi \\
\end{array}
\]  

(2.12)

From this we now proceed to prove

**Proposition 2.13.** For an abelian Hopf algebra \( H \), there is a natural map of modules

\[
\phi : S_2 S^2 V \to S^2 S_2 V
\]

such that the following diagram commutes:

\[
\begin{array}{ccc}
S_2 H & \xrightarrow{S_2(\phi)} & S_2 S^2 H \\
\mu \downarrow & & \downarrow S^2(\mu) \\
H & \xrightarrow{\psi} & S^2 H \\
\end{array}
\]

To prove this we need the following lemmas.

**Lemma 2.14.** Let \( V \) be a (graded) module. Then there exist maps \( \phi' \), \( \phi'' \) of modules such that the following diagram commutes:

\[
\begin{array}{ccc}
S^2 V \otimes S^2 V & \xrightarrow{i \otimes i} & V \otimes V \otimes V \otimes V \\
\phi' \downarrow & & \downarrow 1 \otimes T \otimes 1 \\
S^2(V \otimes V) & \xrightarrow{\rho} & S_2(V \otimes V) \\
\phi'' \downarrow & & \downarrow \phi'' \\
S^2(V \otimes V) & \xrightarrow{i} & V \otimes V \otimes V \otimes V \\
\rho \otimes \rho & & \rho \otimes \rho \\
& & \downarrow \\
& & S_2 V \otimes S_2 V \\
\end{array}
\]

**Proof.** Define \( \hat{T} : V^\otimes 4 \to V^\otimes 4 \) by \( \hat{T}(a \otimes b \otimes c \otimes d) = c \otimes d \otimes a \otimes b \). Then on \( V^\otimes 4 \), we have the identity

\[
(1 \otimes T \otimes 1)(T \otimes T) = \hat{T}(1 \otimes T \otimes 1).
\]
From this, the two composites
\[ S^2 V \otimes S^2 V \xrightarrow{i \otimes i} V^4 \xrightarrow{1 \otimes T \otimes 1} V^4 \]
and
\[ V^4 \xrightarrow{1 \otimes T \otimes 1} V^4 \xrightarrow{p \otimes p} S_2 V \otimes S_2 V \]
factors to give us the maps \( \phi' \) and \( \phi'' \), respectively. \( \square \)

**Lemma 2.15.** For a module \( V \) there exists a map \( \phi \) of modules such that the following cube commutes

\[
\begin{array}{ccc}
S^2 V \otimes S^2 V & \xrightarrow{i \otimes i} & V^4 \\
\downarrow \phi' & & \downarrow \phi \\
S_2 S^2 V & \xrightarrow{1 \otimes T \otimes 1} & S_2(V \otimes V) \\
\downarrow \phi & & \downarrow S_2(p) \\
S^2(V \otimes V) & \xrightarrow{i} & V^4 \\
\downarrow S^2(\rho) & & \downarrow S_2 V \otimes S_2 V \\
S^2 S_2 V & \xrightarrow{i} & S_2 V \otimes S_2 V
\end{array}
\]

**Proof.** The identity \((1 \otimes T \otimes 1)(T \otimes T) = \hat{T}(1 \otimes T \otimes 1)\) from the proof of Lemma (1.7), tells us that the composite
\[ S^2 V \otimes S^2 V \xrightarrow{\phi'} S^2(V \otimes V) \xrightarrow{S^2(\rho)} S^2 S_2 V \]
factors to give us the desired map \( \phi \). The commutativity of the cube now follows from Lemma 2.14, the surjectivity of \( p \), and the injectivity of \( i \). \( \square \)

**Proof of Proposition 2.13.** The diagram (1.7) can be expanded to give
which commutes by (2.10), (2.12), Lemma 2.15, the surjectivity of \( \rho \), and the injectivity of \( i \).

We now pause to give a useful reinterpretation of Proposition 2.13. Let \( A \) be a commutative algebra. Then \( S^2 A \) is a commutative algebra with product

\[
S^2 S^2 A \xrightarrow{\phi} S^2 S^2 A \xrightarrow{S^2(\mu)} S^2 A
\]

and unit

\[
\mathbb{F}_2 \cong S^2(\mathbb{F}_2) \xrightarrow{S^2(\eta)} S^2 A.
\]

\[\textbf{Corollary 2.18.} \text{ For an abelian Hopf algebra } H, \text{ the coproduct } \psi : H \rightarrow S^2 H \]

is a map of commutative algebras.

Also, if \( A \) is a \( \Gamma \)-algebra then \( A \otimes A \) is a \( \Gamma \)-algebra. Moreover, from its definition we have

\[
\gamma_2 T = T \gamma_2.
\]

Thus \( S^2 A \) is also a \( \Gamma \)-algebra.

Now, as an application of Proposition 2.8 we give another interpretation of the Verschiebung for a cocommutative coalgebra \( \Pi \). Fix \( x \in \Pi \). Then \( \psi x \in S^2 \Pi \). By Proposition 2.8 there exists unique \( x \in E_2 \Pi \) and \( \bar{\beta} \in \Pi \) such that

\[
\psi x = v(x) + \sigma(\bar{\beta}).
\]

From this we can set \( \zeta(x) = \bar{\beta} \).

We can use this to record a basic relation on a Hopf \( \Gamma \)-algebra \( H \). Our objective is to give a description of the composite

\[
\Phi I_2 \xrightarrow{\gamma_2} H \xrightarrow{\zeta} \Phi H.
\]

To do so we define a map

\[
h : I_2 \rightarrow H
\]
which fits in the following expansion of (2.7)

\[ \Phi H \xrightarrow{i} S_2H \xrightarrow{\pi} \text{coker } N \xrightarrow{\sigma} \Phi H \]

Here \( \sigma \) is the natural map determined by Proposition 2.8 and the dotted arrow exists in positive degrees by 1 of Definition 1.9.

We can give an explicit description of the map \( h \) as follows: for \( x \in H \) write

\[ \psi x = \Sigma(x' \otimes x'' + x''' \otimes x') + \Sigma y \otimes y \]

then

\[ h(x) = \Sigma x' x''. \quad (2.21) \]

**Proposition 2.22.** For a Hopf \( \Gamma \)-algebra \( H \) the diagram

\[ \Phi I_2 \xrightarrow{\gamma_2} H \]

commutes.

To prove this, we note that since \( i: S^2H \rightarrow H \otimes H \) is a map of \( \Gamma \)-algebras then \( \psi: H \rightarrow S^2H \) is a map of \( \Gamma \)-algebras, by Corollary 2.18 and Definition 1.9. In light of this and Proposition 2.8 we are reduced to proving

**Lemma 2.23.** Let \( A \) be a \( \Gamma \)-algebra and \( \omega \in S^2A \). Write \( \omega = v(\alpha) + \sigma(\bar{\beta}) \) as in Proposition 2.8. Then

\[ \pi(\gamma_2 \omega) = \pi \sigma(\mu(\bar{\alpha})) \]

where \( \bar{\alpha} \in S^2A \) satisfies \( \zeta(\bar{\alpha}) = \alpha \).

**Proof.** Since \( \gamma_2 \) is quadratic, we have

\[ \gamma_2 \omega = \gamma_2 v(\alpha) + \gamma_2 \sigma(\bar{\beta}) + v(\alpha) \cdot \sigma(\bar{\beta}). \]
Using the $I$-algebra map $i : S^2 A \to A \otimes A$ we can compute $\pi(v(\alpha) \cdot \sigma(\vec{b})) = 0$. Also, since $\omega \in I_2$, $\gamma_2 \sigma(\vec{b}) = 0$. We are thus left with computing $\gamma_2 v(\alpha)$. Choose $z \in A \otimes A$ such that it maps to $\alpha$ under $A \otimes A \to E_2 A$ and let $\bar{z}$ be its image in $S_2 A$. Then in $A \otimes A$

$$iv(\alpha) = (1 + T)z$$

so that a computation using (2.19) gives us

$$\gamma_2 iv(\alpha) = (1 + T)\gamma_2 z + z \cdot Tz$$

$$= iv(\gamma) + i\sigma(\mu(\bar{z}))$$

for some $\gamma \in E_2 A$ (in fact $\gamma$ is the image of $\gamma_2 z$). □

We end this section by taking a closer look at the map $\phi$. Let $V$ be a module. Then we have

**Generators of $S^2 S^2 V$:**

$$[x, y] \cdot [z, w],$$

$$\sigma(x) \cdot [y, z],$$

$$\sigma(x) \cdot \sigma(y)$$

for any $x, y, z, w \in V$.

**Generators of $S^2 S_2 V$:**

$$[x \cdot y, z \cdot w],$$

$$\sigma(x \cdot y)$$

for any $x, y, z, w \in V$.

Here $\sigma$ is the map of (2.6).

The effect of

$$\phi : S^2 S^2 V \to S^2 S_2 V$$

is given by

$$[x, y] \cdot [z, w] \to [x \cdot z, y \cdot w] + [x \cdot w, y \cdot z],$$

$$\sigma(x) \cdot [y, z] \to [x \cdot y, x \cdot z],$$

$$\sigma(x) \cdot \sigma(y) \to \sigma(x \cdot y).$$

We can use this to compute the kernel and cokernel of $\phi$. First, we have a map

$$\rho : V^\otimes 4 \to S^2 S^2 V$$

given by

$$a \otimes b \otimes c \otimes d \to [a, b] \cdot [c, d] + [a, c] \cdot [b, d] + [a, d] \cdot [b, c].$$
It is easy to see that 
\[ \phi \theta = 0. \]

Further, we have a factorization

\[
\begin{array}{ccc}
V^\otimes 4 & \xrightarrow{\theta} & S_2 S^2 V \\
& \searrow & \\
& E_4 V & \xleftarrow{\phi}
\end{array}
\]

Here \( E_4 V \) is the fourth exterior power of \( V \) i.e. the cokernel of the composite

\[ (\Phi V) \otimes V^\otimes 2 \xrightarrow{d \otimes 1} V^\otimes 4 \to S_4 V \]

where \( d \) is from (2.6).

**Claim.** The induced map

\[ \overline{\theta} : E_4 V \to \ker \phi \]

is a linear isomorphism.

**Proof.** By naturality of \( \overline{\theta} \) and simplicity of the functor \( E_4 \), \( \overline{\theta} \) is injective, since it is nontrivial. To see surjectivity, we note that \( \overline{\theta} \) is onto when \( \dim V \leq 4 \). Thus, since \( E_4 \) is a polynomial functor of degree \( \leq 4 \) the result follows. \( \square \)

Now, an easy calculation shows

\[ (S_2 V)^* = S^2 V^* \]

and

\[ (S^2 V)^* = S_2 V^*. \]

From this and Lemma 2.15 we have

\[ \phi^* = \phi. \]

Further \( (E_4 V)^* = E_4 V^* \) so that the claim gives us an exact sequence

\[ 0 \to E_4 V \to S_2 S^2 V \xrightarrow{\phi} S^2 S_2 V \to E_4 V \to 0 \]

which is natural as functors of modules. This defines a map

\[ \mathcal{F}_2 \to \text{Ext}_\mathcal{F}^2 (E_4, E_4), \]

where \( \mathcal{F} \) is the category of endofunctors on the category of modules. L. Schwartz has shown (private communication) that this map is an injection.
3. Some explicit computations of homotopy groups

The goal of this section is review the explicit description one can give $\pi_* S^2 V$ and $\pi_* S^2 V$ in terms of $\pi_* V$. From this one can determine all the primary operations for the homotopy of simplicial (co)commutative (co)algebras.

We begin by summarizing the Eilenberg–Zilber theorem as given in [15, 9].

**Theorem 3.1.** Let $V$ and $W$ be two simplicial $\mathbb{F}_2$-modules. Then there exists a unique natural chain map

$$D : N(V) \otimes N(W) \to N(V \otimes W)$$

which is the identity in dimension 0.

Moreover, there exists a natural chain map

$$E : N(V \otimes W) \to N(V) \otimes N(W)$$

such that

$$ED = 1, \quad DE \simeq 1.$$

In [9] it was noticed that since $D$ is necessarily the shuffle map (see [15]) then $D$ possesses a symmetry. This symmetry was exploited by Dwyer to construct higher order versions of $D$ which we now describe.

**Definition 3.2.** For each $k \geq 0$, let

$$\phi_k : N(V) \otimes N(W) \to N(V \otimes W)$$

be the chain map such that for $x \in N(V)$ and $y \in N(W)$

$$\phi_k(x \otimes y) = \begin{cases} x \otimes y, & |x| = k = |y|, \\ 0 & \text{otherwise.} \end{cases}$$

$\phi_k$ is called an admissible map.

Let $T$ denote the switching map for either

$$N(V) \otimes N(W) \to N(W) \otimes N(V)$$

or

$$N(V \otimes W) \to N(W \otimes V).$$

**Theorem 3.3.** Let $V$ and $W$ be simplicial $\mathbb{F}_2$-modules. For each $k \geq 0$ there exists a natural chain map

$$D^k : [N(V) \otimes N(W)]_m \to N(V \otimes W)_{m-k}$$

defined for $m \geq 2k$ and satisfying
1. $D^0 + TD^0 T + \phi_0 = D,$
2. $D^{k+1} + TD^{k+1} T + \phi_{k+1} - \partial D^k + D^k \partial.$

**Remark.** Dwyer showed in [9] that each $D^k$ is unique in a certain sense.

In light of (2.10), a computation of the homotopy of $S_2 V$, for a simplicial module $V$, in terms of $\pi_s V$ would give a complete picture of the primary operator algebra for the homotopy of a simplicial commutative algebra. Such a description is known to exist by [8]. We now proceed to make this description explicit.

Fix a simplicial module $V$. For each $0 \leq i \leq n$ define

$$\Theta_i : N_n V \to N_{n+i} S_2 V$$

by

$$\Theta_i(a) = \mu D^{n-i}(a \otimes a) + \mu D^{n-i-1}(a \otimes a)$$

where the $D^s$ are from Theorem 3.3.

A computation gives us that

$$\partial \Theta_i = \Theta_i \partial.$$

Thus, for $2 \leq i \leq n$, $\Theta_i$ induces a natural map

$$\tilde{\Theta}_i : n_n V \to n_{n+i} S_2 V.$$  (3.6)

Also, the chain map

$$\rho D : N_s V \otimes N_t V \to N_{s+t} S_2 V$$

induces a homomorphism

$$\tilde{m} : \pi_s V \otimes \pi_t V \to \pi_{s+t} S_2 V.$$  (3.7)

Combining the results of [4, 2, 9] we are led to

**Proposition 3.8.** Let $V$ be a fixed simplicial module. Define $W$ to be the graded module with basis

- $\delta_i(x)$ for $x \in \pi_n V$ and $2 \leq i \leq n$,
- $x \cdot y$ for $x \in \pi_s V$ and $y \in \pi_t V$.

Define a submodule $B$ in $W$ with basis

- $\delta_i(x + y) + \delta_i(x) + \delta_i(y)$ for $x, y \in \pi_n V$,
- $x \cdot y + y \cdot x$ for $x \in \pi_s V$ and $y \in \pi_t V$,
- $x \cdot (y + z) + x \cdot y + x \cdot z$ for $x \in \pi_s V$ and $y, z \in \pi_t V$,
- $x \cdot x$ for $x \in \pi_s V$ and $n > 0$. 
Then the map $W \rightarrow \pi_+ S_2 V$ given by
\[ x \cdot y \rightarrow \overline{m}(x \otimes y), \]
\[ \delta_i x \rightarrow \delta_i x \]
is natural and induces a linear isomorphism
\[ W/B \simeq \pi_+ S_2 V. \]

Next, consider the composite
\[ N_n V \xrightarrow{\Theta_i} N_{n+i} S_2 V \xrightarrow{N_*} N_{n+i} S^2 V \]
of chain maps. Here $N$ is the norm map (2.7). This induces a natural map
\[ \sigma_i : \pi_n V \rightarrow \pi_{n+i} S^2 V \]  
(3.10)
for each $0 \leq i \leq n$. Also, the composite
\[ N_n V \otimes N_l V \xrightarrow{D} N_{n+l}(V \otimes V) \xrightarrow{\pi_*} N_{n+l} S_2 V \xrightarrow{N_*} N_{n+l} S^2 V \]
(3.11)
induces the homomorphism
\[ \tau : \pi_n V \otimes \pi_l V \rightarrow \pi_{n+l} S^2 V. \]

The following is given in [12].

**Proposition 3.13.** Let $V$ be a simplicial module. Let $T$ be the graded vector space with basis
\[ \sigma_i(x) \quad \text{for } x \in \pi_n V \text{ and } 0 \leq i \leq n, \]
\[ [x,y] \quad \text{for } x \in \pi_n V \text{ and } y \in \pi_m V, \quad n,m \geq 0. \]

Let $R$ be the submodule of $T$ with basis
\[ [x,y] + [y,x] \quad \text{for } x \in \pi_n V, \quad y \in \pi_m V, \quad n,m \geq 0, \]
\[ [x,y+z] + [x,y] + [x,z] \quad \text{for } x \in \pi_n V, \quad y,z \in \pi_m V, \quad n,m \geq 0, \]
\[ \sigma_i(x+y) + \sigma_i(x) + \sigma_i(y) + \begin{cases} 0, & 0 \leq i < n, \\ [x,y], & i=n \quad \text{for } x,y \in \pi_n V. \end{cases} \]
\[ [x,x] \quad \text{for } x \in \pi_n V, \quad n \geq 0. \]

Then the map $T \rightarrow \pi_+ S^2 V$ defined by
\[ \sigma_i(x) \rightarrow \sigma_i(x), \]
\[ [x,y] \rightarrow \tau(x \otimes y) \]
induces a natural linear isomorphism
\[ T/R \simeq \pi_+ S^2 V. \]
Moreover, if we let
\[ e : \pi_* S^2 V \to \pi_* V \otimes \pi_* V \]
be induced by the composition of chain maps
\[ NS^2 V \xrightarrow{I_2} N(V \otimes V) \xrightarrow{F} NV \otimes NV \]
(see Theorem 3.1) then for \( x \in \pi_n V \quad y \in \pi_m V \quad n, m \geq 0 \)
\[ e([x, y]) = x \otimes y + y \otimes x \]
and for \( x \in \pi_n V \quad 0 < i < n \)
\[ e(\sigma_i(x)) = \begin{cases} 
0, & 0 \leq i < n, \\
x \otimes x, & i = n.
\end{cases} \]

We take a moment to note a corollary given in [12].

**Corollary 3.14.** The effect of the homomorphism
\[ N_* : \pi_* S^2 V \to \pi_* S^2 V \]
is given by
\[ x \cdot y \to [x, y] \]
for \( x \in \pi_n V, \quad y \in \pi_m V, \quad n, m \geq 0, \) and
\[ \delta_i(x) \to \sigma_i(x) \]
for \( x \in \pi_n V, \quad 2 \leq i \leq n. \) Moreover, under the homomorphism (2.6)
\[ \iota_* : \Phi \pi_* V \to \pi_* S^2 V \]
we have
\[ \text{im} \ i_* = \ker N_* . \]

Finally, given \( \Pi \) in \( V \), then for \( x \in \pi_n \Pi \) Proposition 3.13 tells us that
\[ \psi_* x = \Sigma [x', x''] + \Sigma \sigma_l(x Sq^j) \]
which defines the action of the Steenrod operations. From this and Corollary 3.14 we conclude 1 of (1.5). Also, we define the coproduct
\[ A : \pi_* \Pi \to \pi_* \Pi \otimes \pi_* \Pi \]
by \( e\psi_* \) from Proposition 3.13.
4. Reduction in the proof of the main theorem

In this section, we use the results of the previous two sections to state a theorem which allows us to prove Theorem 1.11. First, recall that if $A$ is a simplicial commutative algebra then $S^2A$ is also a simplicial commutative algebra using (2.16). Further if $A$ is a simplicial abelian Hopf algebra then, by Corollary 2.18, the coproduct induces a simplicial algebra map $\psi : A \rightarrow S^2A$. Thus the induced map $\psi_*$ on homotopy groups is a map of $D$-algebras. Hence one approach to proving Theorem 1.11 is to determine $\pi_*S^2A$ as a $D$-algebra based upon Proposition 3.13. For this we now state

**Theorem 4.1.** Let $A$ be a simplicial commutative algebra. Then for the associated simplicial commutative algebra $S^2A$ the following relations hold in the $D$-algebra $\pi_*S^2A$:

(a) For $x \in \pi_nA$, $0 \leq i \leq n$, $2 \leq j \leq n + i$,

$$\delta_j\sigma_i(x) = \sum_{s < 2s} \left( \frac{s - j - 1}{2s - j - 1} \right) \sigma_{i+j-s}\delta_s(x).$$

(b) For $x \in \pi_nA$, $y \in \pi_mA$, $2 \leq j \leq n + m$,

$$\delta_j[x, y] = \begin{cases} 
\sigma_j(x \cdot y) + [x \cdot x, \delta_jy] & \text{if } n = 0, \\
\sigma_j(x \cdot y) + [\delta_jx, y \cdot y] & \text{if } m = 0, \\
\sigma_j(x \cdot y) & \text{otherwise}.
\end{cases}$$

(c) For $x \in \pi_nA$, $y \in \pi_mA$, $0 \leq i \leq n$, $0 \leq j \leq m$,

$$\sigma_i(x) \cdot \sigma_j(y) = \sigma_{i+j}(x \cdot y).$$

(d) For $x \in \pi_nA$, $y, z \in \pi_*A$, $0 \leq i \leq n$,

$$\sigma_i(x) \cdot [y, z] = \begin{cases} 
[x \cdot y, x \cdot z] & \text{if } i = n, \\
0 & \text{otherwise}.
\end{cases}$$

(e) For $x, y, z, w \in \pi_*A$,

$$[x, y] \cdot [z, w] = [x \cdot z, y \cdot w] + [x \cdot w, y \cdot z].$$

With this we now pause to prove Theorem 1.11. First, by Theorems 1.3 and 1.6, $\pi_*H$ is both a $D$-algebra and an $A$-coalgebra. Moreover, $A$ is a map of simplicial commutative algebras by (2.10) and Lemma 2.14. By Theorems 3.1 and 1.3 we conclude $\pi_*H$ is a Hopf $\Gamma$-algebra.

We now proceed to establish 1 and 2 of Definition 1.10. For the remainder of this section we fix $x \in \pi_nH$ and write

$$\psi_*x = \sum_{k} [x_k', x_k''] + \sum_{k} \sigma_k(xSq^k)$$

as in (3.15).
1. The first part is an easy consequence of the fact that $\Delta$ is a map of simplicial commutative algebras. For the second part let $y \in \pi_m H$ and write

$$\psi_* y = \sum [y'_i, y''_i] + \sum \sigma_i(xSq^i).$$

By Theorem 4.1 we have

$$(\psi_* x) \cdot (\psi_* y) \equiv \sum_{s,t} \sigma_s(xSq^s) \cdot \sigma_t(ySq^t)$$

$$\equiv \sum_{i \geq 0} \sum_{s+t=i} \sigma_i(xSq^s \cdot ySq^t)$$

$$\equiv \sum_{i \geq 0} \sigma_i \left( \sum_{s+t=i} xSq^s \cdot ySq^t \right),$$

where, herein after "\equiv" means "equal modulo [ , ]'s". By (3.15) we have

$$\psi_*(x \cdot y) \equiv \sum_{i \geq 0} \sigma_i((x \cdot y)Sq^i).$$

The conclusion follows from Corollary 2.17. □

2. Fix $2 \leq j < n$. By (3.15) we have

$$\psi_* \delta_j(x) \equiv \sum_i \sigma_i((\delta_j x)Sq^i).$$

Next, Theorem 4.1 gives us

$$\delta_j \psi_* x \equiv \sum_k \sigma_j(x'_k \cdot x''_k) + \sum_{s} \delta_j \sigma_s(xSq^s)$$

$$\equiv \sum_k \sigma_j(x'_k \cdot x''_k) + \sum_{s} \sum_{j<2l} \left( \frac{l-s-1}{2l-j-1} \right) \sigma_{j+s-l} \delta_l(xSq^s)$$

$$\equiv \sum_k \sigma_j(x'_k \cdot x''_k) + \sum_{s} \sum_{2l-j<2s} \left( \frac{j-i-1}{j-2i+2s-1} \right) \sigma_i \delta_{j-i+s}(xSq^s)$$

$$\equiv \sum_k \sigma_j(x'_k \cdot x''_k) + \sum_{s} \sigma_i \left( \sum_{2l-j<2s} \left( \frac{j-i-1}{j-2i+2s-1} \right) \delta_{j-i+s}(xSq^s) \right)$$

When $i < j$ we immediately get the third equation. When $i > j$ the expression

$$\binom{m}{r} - \binom{-m + r - 1}{r}$$

gives us the first equation. When $i = j$ we just need to verify

$$\xi_j 2x = \sum x'_k \cdot x''_k$$
which is just a consequence of Proposition 2.22. Finally, combining Theorem 4.1 and Definition 1.1 we get

\[
\delta_n \psi_* x \equiv \sum_k \sigma_n(x'_k \cdot x''_k) + \sum_s \delta_n \sigma_s(x Sq^s) + \sum_{s < t} \sigma_s(x Sq^s) \cdot \sigma_t(x Sq^t)
\]

\[
\equiv \sum_k \sigma_n(x'_k \cdot x''_k) + \sum_s \delta_n \sigma_s(x Sq^s) + \sum_{i \geq 0} \sum_{2s < i} \sigma_i \left( \sum_{2s < i} x Sq^s \cdot x Sq^{i-s} \right)
\]

and so proceeding as before gives us the remaining equations. The conclusion follows from Corollary 2.17. This completes the proof of Theorem 1.11. \(\square\)

We end this section by describing our strategy for proving Theorem 4.1. Our main objective is to reduce our computations to ones in group cohomology. For this we want to follow the classic approach developed by Adem in [1] as made systematic by Dwyer in [9]. To accomplish this we first note that Theorem 4.1 easily follows from:

**Proposition 4.2.** Let \( V \) be a simplicial module. Then the effect of the map

\[ \phi : S^2 S^2 V \rightarrow S^2 S^2 V \]

in homotopy is given by the following:

(a) For \( x \in \pi_n V, 0 \leq i \leq n, 2 \leq j \leq n + i \)

\[ \phi_* \delta_j \sigma_i (x) = \sum_{j < 2s} \binom{s - i - 1}{2s - j - 1} \sigma_{i+j-s} \delta_s (x). \]

(b) For \( x \in \pi_n V, y \in \pi_m V, 2 \leq i \leq n + m \),

\[ \phi_* \delta [x, y] = \begin{cases} \sigma_i (x \cdot y) + [x \cdot x, \delta x] & \text{for } n = 0, \\ \sigma_i (x \cdot y) + [\delta_i x \cdot y \cdot y] & \text{for } m = 0, \\ \sigma_i (x \cdot y) & \text{otherwise}. \end{cases} \]

(c) For \( x \in \pi_n V, y \in \pi_m V, 0 \leq i \leq n, 0 \leq j \leq m \),

\[ \phi_* (\sigma_i (x) \cdot \sigma_j (y)) = \sigma_{i+j} (x \cdot y). \]

(d) For \( x \in \pi_n V, y, z \in \pi_* V, 0 \leq i \leq n \),

\[ \phi_* (\sigma_i (x) \cdot [y, z]) = \begin{cases} [x \cdot y, x \cdot z], & i = n, \\ 0, & \text{otherwise}. \end{cases} \]

(e) For \( x, y, z, w \in \pi_* V \),

\[ \phi_* ([x, y] \cdot [z, w]) = [x \cdot z, y \cdot w] + [x \cdot w, y \cdot z]. \]
We are now reduced to analyzing the natural map $\phi : S^2 S^2 \rightarrow S^2 S^2$ of functors on simplicial vector spaces as induced in homotopy. Unfortunately, this map cannot directly be described in a way that will allow us to use Dwyer's techniques for translating to group cohomology. To get around this, in the next section we use the normalizing map $N : S^2 \rightarrow S^2$ to show that $\phi$ fits into a commuting diagram involving a natural map $\alpha : S^2 S^2 \rightarrow S^2 S^2$. Using the fact that $S^2 S^2 V = (V \otimes^4 D_8)$, where $D_8 < \Sigma_4$ is the dihedral group of order 8, we show that the map $\alpha$ does have the properties that allows us to use Dwyer's techniques and, in fact, easily translate to map of group cohomology that is computable. In Section 6, we briefly review the techniques of group cohomology that we need which will be used in the final section to make our required calculations.

5. Further reductions

In this section, we show how translate information from the map $\phi : S^2 S^2 V \rightarrow S^2 S^2 V$ to a new map $\alpha : S^2 S^2 V \rightarrow S^2 S^2 V$ whose properties make it much more useful for calculations.

We start by recalling from Section 3, that we have the norm map

$$N_V : S^2 V \rightarrow S^2 V$$

whose effect is

$$x \cdot y \rightarrow [x, y].$$

Consider now the maps

$$N_{S^2 V} : S^2 S^2 V \rightarrow S^2 S^2 V$$

and

$$S^2 N_V : S^2 S^2 V \rightarrow S^2 S^2 V.$$ 

It is well known that $S^2 S^2 V = S^2 \Sigma_2 \int \Sigma_2 V$ where $\Sigma_2 \int \Sigma_2$ is the wreath product of $\Sigma_2$ and $\Sigma_2$, i.e. the subgroup of $\Sigma_4$ which fits into the split extension

$$1 \rightarrow \Sigma_2 \times \Sigma_2 \rightarrow \Sigma_2 \int \Sigma_2 \rightarrow \Sigma_2 \rightarrow 1$$

(5.1)

where, in terms of transpositions, we have

$$\Sigma_2 \times \Sigma_2 = \langle (1, 2), (3, 4) \rangle,$$

$$\Sigma_2 = \langle (1, 3)(2, 4) \rangle.$$ 

Moreover, it is well-known that $\Sigma_2 \int \Sigma_2 \simeq D_8$; the dihedral group of order 8. We thus have the identity

$$S^2 S^2 V \simeq S^4 V.$$ 

(5.2)
Lemma 5.3. There exists a natural idempotent map
\[ \alpha : S^{D_S}V \to S^{D_S}V \]
such that the diagram
\[ \begin{array}{ccc}
S_2S^2V & \xrightarrow{N_{\phi V}} & S^{D_S}V \\
\downarrow \phi & & \downarrow \alpha \\
S^2S_2V & \xrightarrow{S^2N_v} & S^{D_S}V 
\end{array} \]
commutes. Explicitly
\[ \alpha = 1 + r(\Sigma_4, D_8)(D_8, \Sigma_4). \]

The proof will follow from the next lemma.

Lemma 5.4. There exists a natural map
\[ \alpha'' : S^2(V \otimes^2) \to (S^2V)^{\otimes 2} \]
such that the diagram
\[ \begin{array}{ccc}
S_2(V \otimes^2) & \xrightarrow{N_{\phi^2 V}} & S^2(V \otimes^2) \\
\downarrow \phi'' & & \downarrow \alpha'' \\
(S_2V)^{\otimes^2} & \xrightarrow{(N_v)^{\otimes 2}} & (S^2V)^{\otimes 2} 
\end{array} \]
commutes. Here \( \phi'' \) is the map of Lemma 2.14. Indeed, we can take
\[ \alpha'' = \varepsilon t(\Sigma_2, \Sigma_2 \times \Sigma_2) \]
where the transfer is associated with the diagonal \( \Sigma_2 \to \Sigma_2 \times \Sigma_2 \) and \( \varepsilon \) is the isomorphism induced by \( 1 \otimes T \otimes 1 : V^{\otimes 4} \to V^{\otimes 4} \).

Proof. First, we have commuting diagrams
\[ \begin{array}{ccc}
V^{\otimes 4} & \xrightarrow{1+(1,3)(2,4)} & V^{\otimes 4} \\
\downarrow \rho_{V^{\otimes 2}} & & \downarrow i_{V^{\otimes 2}} \\
S_2(V^{\otimes 2}) & \xrightarrow{N_{V^{\otimes 2}}} & S^2(V^{\otimes 2}) 
\end{array} \]
An easy computation shows that the diagram

\[
\begin{array}{ccc}
V \otimes^4 & \xrightarrow{1+(1,2)(1+(3,4))} & V \otimes^4 \\
\downarrow_{\iota_2} & & \downarrow_{\iota_2} \\
(S^2V) \otimes^2 & \xrightarrow{N_2} & (S^2V) \otimes^2.
\end{array}
\]

commutes. Consider now the map

\[
S^2(V \otimes^2) \xrightarrow{\iota_2} V \otimes^4.
\]

In the group ring $F_2[\Sigma_4]$, we have the identity

\[
(2,3)(1+(1,3))(1,3)(2,4) = (1+(1,2))(2,3)(1,3)(2,4)
\]

\[
= (1+(1,2))(2,3)(1,3)(2,4)
\]

\[
= ((1,2)(3,4) + (3,4))(2,3).
\]

This shows that the image of the above map is invariant under the action of \((1,2), (3,4)\). We thus have a commuting diagram

\[
\begin{array}{ccc}
S^2(V \otimes^2) & \xrightarrow{\iota_2 \otimes^2} & V \otimes^4 \\
\downarrow & & \downarrow \\
(S^2V) \otimes^2 & \xrightarrow{N_2} & V \otimes^4
\end{array}
\]

defining $x''$. 

Combining these four diagrams and Lemma 2.14 gives us a cube

\[
\begin{array}{ccc}
V^\otimes 4 & \rightarrow & V^\otimes 4 \\
\downarrow & & \downarrow \\
S_2(V^\otimes 2) & \rightarrow & S^2(V^\otimes 2) \\
\downarrow & & \downarrow \\
(S_2V)^\otimes 2 & \rightarrow & (S^2V)^\otimes 2
\end{array}
\]

from which our desired commutative diagram results. The identification of \( \alpha'' \) follows from our construction and the definition of transfer. □

**Proof of Lemma 5.3.** Consider the composite

\[
S^2S^2V \xrightarrow{S^2 i_V} S^2(V^\otimes 2) \xrightarrow{\alpha''} (S^2V)^\otimes 2.
\]

From Lemma 5.4 and a computation we have

\[
\alpha''(1,2)(3,4) = e(1,2)(3,4) t(\Sigma_2, \Sigma_2 \times \Sigma_2) = (1,3)(2,4) \alpha''.
\]

Thus \((1,3)(2,4)\alpha'' S^2(i_V) = \alpha'' S^2(i_V)\). Hence, we have a diagram

\[
\begin{array}{ccc}
S^2S^2(V) & \xrightarrow{S^2 i_V} & S^2(V^\otimes 2) \\
\downarrow & & \downarrow \\
(S^2S^2(V)) & \xrightarrow{t_{S^2V}} & (S^2V)^\otimes 2.
\end{array}
\]

By Lemmas 2.15 and 5.4 our desired diagram commutes. From this and the identity

\[(2,3)(1,3) - (1,3)(1,2)\]

we arrive at the commuting diagram

\[
\begin{array}{ccc}
S^2S^2(V) & \rightarrow & V^\otimes 4 \\
\downarrow & & \downarrow \\
S^2S^2(V) & \rightarrow & V^\otimes 4
\end{array}
\]

Clearly \(1, (2,3), (1,3)\) are coset representatives for \(D_8 \times \Sigma_4\). Also \((2,3) + (1,3)^2 = (1,3)(1,2) + (2,3)(1,2)\) from the above and the identity \((1,3)(2,3) = (2,3)(1,2)\). Hence \(\alpha^2 = \alpha\). □
Corollary 5.5. The following cube commutes:

\[ \xymatrix{ S_2S^2V \ar[r] \ar[d]_{\phi} & S^2S^2V \ar[d]^a \ar[r] & S^2(V^\otimes) \ar[d]^{\phi^\sigma} \ar[r] & S^2(V^\otimes) \ar[d]^{x^\sigma} \ar[r] & S^2S^2V \ar[d]_{(S_2V)^\otimes} \ar[r] & (S^2V)^\otimes \ar[d] \ar[r] & \cdot \ar[r] & \cdot } \]

Proof. This easily follows from Lemmas 2.15, 5.3, 5.4, and naturality. \qed

Note. The effect of the map

\[ \alpha : S^{D_8}V \to S^{D_8}V \]

on elements is

\[
\begin{align*}
[[x, y], [z, w]] &\to [[x, z], [y, w]] + [[x, w], [y, z]], \\
[\sigma(x), [y, z]] &\to [[x, y], [x, z]], \\
[\sigma(x), \sigma(y)] &\to \sigma[x, y], \\
\sigma[x, y] &\to \sigma[x, y],
\end{align*}
\]

from which we easily verify idempotence. We further note that the module of natural maps

\[ (-)^{D_8} \to (-)^{D_8} \]

on the category of \( \Sigma_4 \)-modules has as basis the set \{1, \alpha\}. In light of this, Lemma 5.3 should not be surprising.

Now, by Proposition 3.13 we have

Proposition 5.6. The following are generators of \( \pi_*S^2S^2V \):

(a) \( \sigma_j\sigma_i(x) \) for \( x \in \pi_nV, \ 0 \leq i \leq n, \ 0 \leq j \leq n + i \).

(b) \( \sigma [x, y] \) for \( x \in \pi_nV, \ y \in \pi_mV, \ 0 \leq i \leq n + m \).

(c) \( [\sigma_i(x), \sigma_j(y)] \) for \( x \in \pi_nV, \ y \in \pi_mV, \ 0 \leq i \leq n, \ 0 \leq j \leq m \).

(d) \( [\sigma_i(x), [y, z]] \) for \( x \in \pi_nV, \ y, z \in \pi_nV, \ 0 \leq i \leq n \).

(e) \( [x, y], [z, w] \) for \( x, y, z, w \in \pi_nV \).

By Corollary 3.14, the effect of the map

\[ (N_{S^2V})_* : \pi_*S_2^2V \to \pi_*S^2S_2V \]
is given by
\[ \delta_j \sigma_i(x) \to \sigma_j \sigma_i(x), \]
\[ \delta_j [x, y] \to \sigma_j [x, y], \]
\[ \sigma_i(x) \cdot \sigma_j(y) \to [\sigma_i(x), \sigma_j(y)], \]
\[ \sigma_i(x) \cdot [y, z] \to [\sigma_i(x), [y, z]], \]
\[ [x, y] \cdot [z, w] \to [[x, y], [z, w]]. \]

Also, the effect of the map
\[ (S^2 N_V)_\ast : \pi_* S^2 S^2 V \to \pi_* S^2 S^2 V \]
is given by
\[ \sigma_i \delta_j(x) \to \sigma_j \sigma_i(x), \]
\[ \sigma_i(x \cdot y) \to \sigma_i [x, y], \]
\[ [\delta_i(x), \delta_j(y)] \to [\sigma_i(x), \sigma_j(y)], \]
\[ [\delta_i(x), y \cdot z] \to [\sigma_i(x), [y, z]], \]
\[ [x \cdot y, z \cdot w] \to [[x, y], [z, w]]. \]

Further, by Proposition 3.13, the effect of the map
\[ (S_2 \iota_V)_\ast : \pi_* S^2 S^2 V \to \pi_* (S^2 V) \otimes 2 \]
is given by
\[ \delta_j \sigma_i(x) \to \begin{cases} \delta_j (x \otimes x), & i = |x|, \\ 0, & \text{otherwise}, \end{cases} \]
\[ \delta_j [x, y] \to \delta_j (x \otimes y + y \otimes x), \]
\[ \sigma_i(x) \cdot \sigma_j(y) \to \begin{cases} (x \otimes x) \cdot (y \otimes y), & i = |x|, j = |y|, \\ 0, & \text{otherwise}, \end{cases} \]
\[ \sigma_i(x) \cdot [y, z] \to \begin{cases} (x \otimes x) \cdot (y \otimes z + z \otimes y), & i = |x|, \\ 0, & \text{otherwise}, \end{cases} \]
\[ [x, y] \cdot [z, w] \to (x \otimes y + y \otimes x) \cdot (z \otimes w + w \otimes z). \]

Also, the effect of the map
\[ (i_{S_2 N_V})_\ast : \pi_* S^2 S^2 V \to \pi_* (S_2 V) \otimes 2 \]
is given by
\[ \sigma_i \delta_j(x) \to \begin{cases} \delta_j (x \otimes x) \otimes \delta_j(x), & i = |x| + j, \\ 0, & \text{otherwise}, \end{cases} \]
\[
\sigma_i(x \cdot y) = \begin{cases} 
(x \cdot y) \otimes (x \cdot y), & i = |x| + |y|, \\
0 & \text{otherwise},
\end{cases}
\]

\[
[\delta_i(x), \delta_j(y)] \rightarrow \delta_i(x) \otimes \delta_j(y) + \delta_j(x) \otimes \delta_i(x),
\]

\[
[\delta_i(x), y \cdot z] \rightarrow \delta_i(x) \otimes (y \cdot z) + (y \cdot z) \otimes \delta_i(x),
\]

\[
x \cdot y \cdot z \cdot w \rightarrow (x \cdot y) \otimes (z \cdot w) + (z \cdot w) \otimes (x \cdot y).
\]

From this we conclude that the map

\[
(S^2N_V)_* \otimes (i_{s_2})_* : \pi_* S^2 S_2 V \rightarrow \pi_* S^2 S^2 V \otimes \pi_*(S_2 V)^{\otimes 2}
\]

is injective. We are thus reduced, by Corollary 5.5, to computing, in homotopy, the maps induced by \( \alpha \) and \( \phi'' \). For this we have

**Proposition 5.7.** Let \( V \) be a simplicial module. Then the effect of

\( \alpha_* : \pi_* S^2 S^2 V \rightarrow \pi_* S^2 S^2 V \)

is given by

(a) For \( x \in \pi_n V, \ 0 \leq i \leq n, \ 0 \leq j \leq n + 1, \)

\[
\alpha_* \sigma_i \sigma_j (x) = \sum_{s \leq i} \binom{s-i-1}{2s-j-1} \sigma_{i+j-s} \sigma_s (x).
\]

(b) For \( x \in \pi_n V, \ y \in \pi_m V, \ 0 \leq i \leq n + m, \)

\[
\alpha_* \sigma_i [x, y] = \sigma_i [x, y].
\]

(c) For \( x \in \pi_n V, \ y \in \pi_m V, \ 0 \leq i \leq n, \ 0 \leq j \leq m, \)

\[
\alpha_* [\sigma_i (x), \sigma_j (y)] = \sigma_{i+j} [x, y].
\]

(d) For \( x \in \pi_n V, \ y, z \in \pi_* V, \ 0 \leq i \leq n, \)

\[
\alpha_* [\sigma_i (x), [y, z]] = \begin{cases} 
[[x, y], [x, z]], & \ i = n, \\
0 & \text{otherwise}. 
\end{cases}
\]

(e) For \( x, y, z, w \in \pi_* V, \)

\[
\alpha_* [[x, y], [z, w]] = [[x, z], [y, w]] + [[x, w], [y, z]].
\]

**Proposition 5.8.** Let \( V \) be a simplicial module. Then the effect of

\( \phi''_* : \pi_* S_2 (V^{\otimes 2}) \rightarrow \pi_*(S_2 V)^{\otimes 2} \)

is given by

(a) For \( x \in \pi_n V, \ y \in \pi_n V, \ y \in \pi_m V, \ 2 \leq j \leq n + m, \)

\[
\phi''_* \delta_j (x \otimes y) = \begin{cases} 
\delta_x \otimes (y \cdot y), & m = 0, \\
(x \cdot x) \otimes \delta_y, & n = 0, \\
0 & \text{otherwise}. 
\end{cases}
\]
We will actually prove a much more general result then Proposition 5.8. To state it we first need the following set up.

Let $V$ and $W$ be modules. Then the map

$$1 \otimes T \otimes 1 : V \otimes W \otimes V \otimes W \rightarrow V \otimes V \otimes W \otimes W$$

induces

$$\tilde{\phi}'' : S_2(V \otimes W) \rightarrow S_2 V \otimes S_2 W.$$

Following the proof of Lemma 5.4 verbatim gives us

**Lemma 5.9.** There exists a map

$$\tilde{x}'' : S^2(V \otimes W) \rightarrow S^2 V \otimes S^2 W$$

such that the diagram

$$\begin{array}{ccc}
S_2(V \otimes W) & \xrightarrow{N_{V \otimes W}} & S^2(V \otimes W) \\
\phi'' \downarrow & & \downarrow \tilde{x}'' \\
S_2 V \otimes S_2 W & \xrightarrow{N_{V \otimes W}} & S^2 V \otimes S^2 W
\end{array}$$

commutes. Indeed we can take

$$\tilde{x}'' = \epsilon t(\Sigma_2, \Sigma_2 \times \Sigma_2)$$

as in Lemma 5.4.

**Proposition 5.10.** Let $V$ and $W$ be simplicial modules. Then the effect of

$$\tilde{\phi}'' : \pi_* S_2(V \otimes W) \rightarrow \pi_* S_2 V \otimes S_2 W$$

is given by

(a) For $x \in \pi_n V$, $y \in \pi_m V$, $2 \leq j \leq n + m$

$$\tilde{\phi}_* \delta_j (x \otimes y) = \begin{cases} 
\delta_j x \otimes (y \cdot y), & m = 0, \\
(x \cdot x) \otimes \delta_j y, & n = 0, \\
0 & \text{otherwise}.
\end{cases}$$

(b) For $x, z \in \pi_* V$, $y, w \in \pi_* W$,

$$\tilde{\phi}_* [(x \otimes y) \cdot (z \otimes w)] = (x \cdot z) \otimes (y \cdot w).$$
Clearly Proposition 5.10 implies Proposition 5.8. Finally, Proposition 4.2 follows from Lemmas 5.3 and 5.4, Propositions 5.7 and 5.8. The proof of Propositions 5.7 and 5.10 will be given in the last section.

6. Recollections on group cohomology

In this section, we gather the tools necessary for proving Propositions 5.7 and 5.10. The key is the following theorem found in [9].

**Theorem 6.1.** Given a simplicial module $V$ and a subgroup $G \leq \Sigma$ there exists a natural homomorphism

$$
\psi^G : \pi_i^G V \to \bigoplus_{0 \leq k} H^k(G; \pi_{i+k} V^\otimes m)
$$

such that for a subgroup $H \leq G$

$$
\psi^H \psi^G = \psi^G \psi^H, \quad \tau^H \tau^G = \tau^G \tau^H.
$$

The usefulness of the map of Theorem 6.1 is now made precise by the following.

**Proposition 6.2.** For any simplicial module $V$, the natural homomorphism of Theorem 6.1 is injective for the group $\Sigma_2$.

*Note.* Theorem 6.1 is different from Proposition (5.1) in [9] in that the source involves $S^G$. One can obtain our map directly from Dwyer’s construction by ignoring the introduction of the norm element. Furthermore, Proposition 6.2 is a strengthening of Lemma 5.11 in [9] and, as such, we present its modified proof making liberal reference to section 5 of [9].

**Proof.** We first prove the result for $V = K(n)$ the Eilenberg–MacLane object. From [9] we have

$$
N_s(K(n) \otimes K(n)) = \begin{cases} 
0, & s < n, \\
\text{nonzero}, & n \leq s \leq 2n, \\
0, & 2n < s.
\end{cases}
$$

Let $C$ be the $\Sigma_2$-chain complex such that

$$
C_s = \begin{cases} 
F_2[\Sigma_2](x_s), & n < s \leq 2n, \\
F_2(y), & n = s, \\
0 & \text{otherwise}.
\end{cases}
$$

If we write $\Sigma_2 = \{1, T\}$, then the differential $\partial$ on $C$ is given by

$$
\partial x_{s+1} = (1 + T)x_s, \quad n < s < 2n,
$$

$$
\partial x_{n+1} = y.
$$
Write $\pi_n K(n) = \mathbb{F}_2[a]$ and define a map

$$f : C \to N(K(n) \otimes K(n))$$

by

$$x_s \to D^{2n-s}(a \otimes a),$$

$$y \to \phi_n(a \otimes a).$$

By Proposition 3.3 of [9] this is a map of $\Sigma_2$-chain complexes. Moreover, it is a quasi-isomorphism. Let $F$ be the functor $H^0(\Sigma_2; -)$. We wish to compute $H_n F(C) \to \mathcal{R}^{-n} F(C)$.

To do so define the complex $\hat{C}$ by

$$\hat{C}_s = \begin{cases} \mathbb{F}_2[\Sigma_2](x_s), & s \leq 2n, \\ 0, & \text{otherwise} \end{cases}$$

with differential $\hat{d}$ given by $\hat{d}x_{s+1} = (1 + T)x_s$. This is clearly a free $\Sigma_2$-chain complex and the map

$$C \to \hat{C}$$

given by

$$x_s \to \hat{x}_s,$$

$$y \to (1 + T)\hat{x}_n$$

is clearly a quasi-isomorphism. Thus, $\mathcal{R}F(C) = F(\hat{C})$ and an easy calculation gives that

$$H_s F(C) \to \mathcal{R}^{-s} F(C)$$

is an injection for all $s$.

To obtain the general case, we first take $V$ so that $NV$ is bounded above. Then we have a weak equivalence

$$\bigoplus_{x} K(n_x) \to V.$$ 

Thus it suffices to show that if $\Psi^{\Sigma_2}$ is injective for $W_1$ and $W_2$ then it is injective for $W_1 \oplus W_2$. First, we have a decomposition of

$$N((W_1 \oplus W_2) \otimes (W_1 \oplus W_2))$$

as

$$N(W_1 \otimes W_1) \oplus N(W_2 \otimes W_2) \oplus N((W_1 \otimes W_2) \oplus (W_2 \otimes W_1)).$$

Since the last summand is $\Sigma_2$-free and since $\Psi^{\Sigma_2}$ respects this decomposition, injectivity follows. A limit argument completes the proof. $\square$
Corollary 6.3. For any simplicial module $V$, $\Psi^G$ is injective for $G = \Sigma_2 \times \Sigma_2$ and $G = \Sigma_2 \cap \Sigma_2$.

The proof is just a specialized adaptation of Lemmas 5.12 and 5.13 of [9].

We now review some basic tools from the cohomology of groups (see [5] or [11]). Consider the extension of finite groups

$$K \to G \to Q.$$ 

Let $M$ be a $G$-module. Then we have a first quadrant spectral sequence

$$E_2^{*,*} = H^*(Q; H^*(K; M)) \Rightarrow H^*(G; M).$$

Here $H^*(K; M)$ is a $Q$-module since we have the functor

$$H^0(K; -): \mathcal{C}_G \to \mathcal{C}_Q.$$ 

To make this spectral sequence useful we have

Lemma 6.5. Given a diagram

\[
\begin{array}{ccc}
K & \to & G \\
\downarrow & & \downarrow \\
K' & \to & G'
\end{array}
\]

whose rows are extensions then the induced map

$$H^*(Q'; H^*(K'; M)) \to H^*(Q; H^*(K; M))$$

is a map of spectral sequences for a $Q$-module $M$. Moreover, the induced map on $E_\infty$ is compatible with

$$H^*(G'; M) \to H^*(G; M).$$

Further, if the vertical maps are injective, then the map

$$H^*(Q; H^*(K; M)) \to H^*(Q'; H^*(K'; M))$$

induced from the associated transfers, becomes a map of spectral sequences. Again, the induced map on $E_\infty$ is compatible with the associated transfer,

$$H^*(G; M) \to H^*(G'; M).$$

Let $H, K$ be subgroups of a finite group $G$. A double coset representation of $G$ with respect to $H$ and $K$ is a subset $S \subset G$ such that

$$G = \bigcup_{a \in S} H\sigma K$$
and is minimal among all such subsets. Next, if \( x \in G \) and \( J \leq G \) define the conjugation map
\[
c_x : J \rightarrow xJx^{-1}
\]
by \( c_x(u) = xux^{-1} \).

**Proposition 6.6.** Let \( S \) be a double coset representation of \( G \) with respect to \( H \) and \( K \) and let \( M \) be a \( G \)-module. Then for \( x \in H^*(K;M) \)
\[
r(G,H)t(K,G)(x) = \sum_{x \in S} t(H \cap xKx^{-1}, H)r(xKx^{-1}, H \cap xKx^{-1})c_x(z)
\]
holds in \( H^*(H;M) \).

Finally, we recall (see [5, p.256]) that if \( u \in H^*(K;M) \) and \( x \in H^*(G;M) \) then
\[
t(K,G)[(r(G,K)u) \cdot x] = u \cdot t(K,G)x.
\]

### 7. Final proof

In this section, we prove Propositions 5.7 and 5.10 using the methods of the previous section. First, we need some basic results to facilitate our computations.

Let \( K(n) \) be the Eilenberg-MacLane module so that \( \pi_*K(n) = F_2\langle a \rangle \) where \( |a| = n \geq 0 \). Then by Proposition 3.13
\[
\pi_*S^2K(n) \cong \begin{cases} F_2\langle \sigma_i(a) \rangle, & * = n + i, \ 0 \leq i \leq n, \\ 0, & \text{otherwise}. \end{cases}
\]
Also \( H^*(\Sigma_2;F_2) = F_2[w] \) where \( w \) is dual to the generator \( H_1(\Sigma_2;F_2) \cong F_2 \). We then have

**Proposition 7.1.** Under the homomorphism
\[
\Psi^{\Sigma_2} : \pi_*S^2K(n) \rightarrow H^*(\Sigma_2;F_2)
\]
of Theorem 6.1
\[
\Psi^{\Sigma_2}\sigma_i(a) = w^{2n-i}
\]
for all \( 0 \leq i \leq n \).

**Proof.** This follows easily from Proposition 6.2. \( \square \)

Now, take \( K(m) \) so that \( \pi_*K(m) \cong F_2\langle b \rangle \) where \( |b| = m \geq 0 \).
Proposition 7.2. Let\( M \) be the \( \Sigma_2 \)-submodule of \( \pi_*(K(n) \times K(m))^{\otimes 2} \) generated by \( a \otimes b \). Then
\[
H^i(\Sigma_2; M) = \begin{cases} 0, & i > 0, \\ \mathbb{F}_2(\psi_{\Sigma_2}[a, b]), & i = 0. \end{cases}
\]

Proof. For \( i > 0 \) this just follows from the fact that \( M \) is a free \( \Sigma_2 \)-module. For \( i = 0 \) we note that under the projections
\[
S^2(K(n) \times K(m)) \to S^2K(n), \\
S^2(K(n) \times K(m)) \to S^2K(m),
\]
\([a, b]\) projects to 0 in homotopy. Hence, by naturality and Proposition 6.2 the result follows.

Proposition 7.3. Consider the extension
\[
\Sigma_2 \times \Sigma_2 \hookrightarrow D_8 \twoheadrightarrow \Sigma_2.
\]
Then for a simplicial module \( V \)
\[
H^*(D_8; \pi_* V^{\otimes 4}) \simeq H^*(\Sigma_2; H^*(\Sigma_2; \pi_* V^{\otimes 2})^{\otimes 2}).
\]
Moreover, we have a factorization

\[
\begin{array}{ccc}
\pi_* S^0 D_8 V & \xrightarrow{\psi_{D_8}} & H^*(D_8; \pi_* V^{\otimes 4}) \\
\downarrow \pi_* S^2 S^2 V & & \downarrow \pi_* S^0 D_8 V \\
H^*(\Sigma_2; \pi_* (S^2 V)^{\otimes 2}) & \xrightarrow{H^*(\Sigma_2; \psi_{\Sigma_2}^{\otimes 2})} & H^*(\Sigma_2; H^*(\Sigma_2; \pi_* V^{\otimes 2})^{\otimes 2}) \\
\end{array}
\]

Proof. Define functors
\[
F_1 : \mathcal{C}_{D_8} \to \mathcal{C}_{\Sigma_2}
\]
and
\[
F_2 : \mathcal{C}_{\Sigma_2} \to \text{(modules)}
\]
by \( F_1 = H^0(\Sigma_2 \times \Sigma_2; -) \) and \( F_2 = H^0(\Sigma_2; -) \). Then \( F_1 \) preserves injectives and \( F_2 \circ F_1 = H^0(D_8; -) \). So by (5.8) of [9],
\[
\mathcal{A}(F_2 \circ F_1) \simeq \mathcal{A}F_2 \circ \mathcal{A}F_1.
\]
Thus, it suffices to compute $H_*(\mathcal{A}F_2 \circ \mathcal{A}F_1)$ for $NV^{\otimes 4}$. Since we have an equivariant equivalence

$$NV^{\otimes 4} \rightarrow (\pi_* V)^{\otimes 4}$$

and since $\mathcal{A}F_1$ is $\Sigma_2$-equivalent to $\mathcal{A}F_2 \otimes \mathcal{A}F_2$ we conclude that we have a $\Sigma_2$-equivalence

$$\mathcal{A}F_1(NV^{\otimes 4}) \rightarrow H^*(\Sigma_2; \pi_* V^{\otimes 2})^{\otimes 2}$$

so by (5.10) of [9]

$$H_*(((\mathcal{A}F_2 \circ \mathcal{A}F_1)(NV^{\otimes 4})) \simeq H^*(\Sigma_2; H^*(\Sigma_2; \pi_* V^{\otimes 2})^{\otimes 2}).$$

The identification of $\Psi^{D_8}$ follows, again, from (5.8) of [9].

**Note.** The identification in Proposition 7.3 can also be worded to say that the spectral sequence (6.4) collapses at the $E^2$-term. We also note that this identification gives us a choice of representatives for the generators for the cohomology of $D_8$, but we will see that in most cases the spectral sequence (6.4) has only one nontrivial column or row at $E^2$, forcing our hand.

Before proving Proposition 5.7, we note that by Lemma 5.3 and Proposition 6.1 we have

$$x_* \Psi^{D_8} = \Psi^{D_8} x_*.$$  \hfill (7.4)

Also, combining Lemma 5.3 and Proposition 6.6, we have

**Proposition 7.5.** Let $\Delta \leq D_8$ be the subgroup $\langle 2, 3 \rangle D_8 \langle 2, 3 \rangle \cap D_8$. Then the map

$$x_* : H^*(D_8; \pi_* V^{\otimes 4}) \rightarrow H^*(D_8; \pi_* V^{\otimes 4})$$

satisfies the identity

$$x_* = t(\Delta, D_8) c_{\langle 2, 3 \rangle} r(D_8, \Delta).$$

Now, we proceed to prove Proposition 5.7. To do so we exploit naturality using the representability of homotopy (see [12]) and reduce to universal examples. To this end we fix the following throughout:

- $\pi_* K(m) = \mathbb{F}_2 \langle a \rangle$, \hspace{1em} $|a| = m$,
- $\pi_* K(n) = \mathbb{F}_2 \langle b \rangle$, \hspace{1em} $|b| = n$,
- $\pi_* K(p) = \mathbb{F}_2 \langle d \rangle$, \hspace{1em} $|d| = p$,
- $\pi_* K(q) = \mathbb{F}_2 \langle e \rangle$, \hspace{1em} $|e| = q$,

where $m, n, p, q \geq 0$. 

Proof of Proposition 5.7. part (a): First, since $\Delta = \Sigma_2 \times \Sigma_2$, $H^*(\Delta; \mathbb{F}_2) \simeq \mathbb{F}_2[v_1, v_2]$ where $v_1, v_2 \in H^1(\Delta; \mathbb{F}_2)$ is dual to the elements of $H_1(\Delta; \mathbb{F}_2)$ associated to the generators of $\Delta$. We now summarize a result in [9].

Proposition 7.6. There exist elements $x, y \in H^1(D_8; \mathbb{F}_2)$ and $z \in H^2(D_8; \mathbb{F}_2)$ such that
1. $H^*(D_8; \mathbb{F}_2) = \mathbb{F}_2[x, y, z]/(xy)$,
2. $r(D_8, \Delta)x = v_2$,
3. $r(D_8, \Delta)y = 0$,
4. $r(D_8, \Delta)z = v_1(v_1 + v_2)$.

Proof. (1) Follows from Proposition 7.3 plus a determination of extensions which is performed in [1].
(2) Is another calculation done in [1].
(3) Is a computation performed in [9].

Proposition 7.7. Under the homomorphism
$$\psi^{D_8}: \pi_* S^{D_8} K(m) \rightarrow H^*(D_8; \mathbb{F}_2),$$
$$\psi^{D_8}(\Sigma) = x^{N+i-j}z^{m-i}.$$

Proof. As shown in [9], under the identification of Proposition 7.3, $x'$ is the element $w'$ in $H^*(\Sigma_2, H^0(\Sigma_2; \pi_2 m K(m)^{\otimes 2})^{\otimes 2})$ and $z'$ is the element $w' \otimes w'$ in $H^0(\Sigma_2; H^0(\Sigma_2; \pi_2 m K(m)^{\otimes 2})^{\otimes 2})$. The result now follows from Propositions 7.1 and 7.3.

Before getting to our main computation, we need

Lemma 7.8. Let $N \in \mathbb{Z}$ and $a \geq r \geq 0$. Then
$$\sum_{0 \leq l \leq r} \binom{r}{l} \binom{N}{s-l} = \binom{N+r}{s}.$$

Proof. This follows from an easy induction on $r$ using the general Pascal's identity.

Now, combining (7.4), Propositions 7.5, 7.6, and 7.1 we have
$$\psi^{D_8}(x, \sigma_j \sigma_i(a)) = \alpha_s(x^{m+i-j}z^{m-i}).$$
\[= t(\Delta, D_8)c_{(2,3)}(x^iz^j)\]
\[= t(\Delta, D_8)c_{(2,3)}(v_1v_2'((v_1 + v_2)'))\]
\[= t(\Delta, D_8)(v_1v_2'(v_1 + v_2)')\]
\[= t(\Delta, D_8) \left( \sum_{0 \leq k \leq t} \binom{t}{k} v_1^{s+t-k} v_2'^{t+k} \right). \quad (7.9)\]

Here we have the identity \(s = m + i - j\) and \(t = m - i\). We have also slipped in \(c_{(2,3)}v_1 = v_2\).

Now, by (6.7) and Proposition 7.6, (7.9) becomes
\[= \sum_{0 \leq k \leq t} \binom{t}{k} x^{t+k} \left( \sum_{0 \leq l < s+t-k} \binom{s+t-k-l-1}{l} x^{s+t-k-2l} z^l \right)\]
\[= \sum_{0 \leq l < s+t} \sum_{0 \leq k \leq t} \binom{t}{k} \binom{s+t-k-l-1}{l} x^{s+t-2l} z^l, \quad (7.10)\]

where the last equality follows since \(k < s + t - 2l\). Now, for each \(k\)
\[\begin{pmatrix} s+t+k-l-1 \\ l \end{pmatrix} = \begin{pmatrix} s+t-k-l-1 \\ s+t-k-2l-1 \end{pmatrix} = \begin{pmatrix} -l-1 \\ s+t-2l-k-1 \end{pmatrix}.\]

Applying Lemma 7.8, we obtain
\[\sum_{0 \leq k \leq t} \binom{t}{k} \begin{pmatrix} s+t+k-l-1 \\ l \end{pmatrix} = \begin{pmatrix} t-l-1 \\ s+t-2l-1 \end{pmatrix}.\]

Thus, (7.10) becomes
\[\sum_{0 \leq l < s+t} \binom{t-l-1}{s+t-2l-1} x^{s+t-2l} z^l = \sum_{j < 2s} \binom{s-i-1}{2s-j-1} x^{n-i-j+2s} z^{n-s} \]
upon letting \(l = n - s\).

Combining Proposition 6.1, (7.4), Propositions 7.5 and 7.7 we arrive at our desired result.

**Proof of Proposition 5.7.** parts (b) and (c): As before, it is sufficient to prove it for the case \(V = K(m) \times K(n)\).
Let $N$ be the $\Sigma_4$-submodule of $\pi_*(K(m) \times K(n))^\otimes 4$ generated by $a \otimes a \otimes b \otimes b$. As such $N$ is a direct summand of $\pi_*(K(m) \times K(n))^\otimes 4$ as a $\Sigma_4$-module. Further, as a $D_8$-module

$$N = N_1 \otimes N_2,$$

where $N_1$ is generated by $a \otimes a \otimes b \otimes b$ and $N_2$ is generated by $a \otimes b \otimes a \otimes b$. Now, writing the extension of (5.1) as

$$B \rightarrow D_8 \rightarrow \Sigma_2,$$

where $B = \langle (1,2),(3,4) \rangle \simeq \Sigma_2 \times \Sigma_2$, then $N$ is a direct sum of two trivial $B$-modules. Thus by the Kunneth theorem

$$H^*(B; N_1) \simeq H^*(B; \mathbb{F}_2) \oplus H^*(B; \mathbb{F}_2) \simeq \mathbb{F}_2[\zeta', \zeta_2'] \oplus \mathbb{F}_2[\zeta'', \zeta_2''].$$

Here $\Sigma_2$ acts by exchanging summands, which is a free $\Sigma_2$-action. Hence, (6.4) implies that

$$H^*(D_8; N_1) \simeq H^0(\Sigma_2; H^*(B; N_1)) \simeq \mathbb{F}_2[\zeta_1, \zeta_2],$$

where $\zeta_1$ corresponds to $\zeta'_1 \oplus \zeta''_1$ and $\zeta_2$ corresponds to $\zeta'_2 \oplus \zeta''_2$, $|\zeta_1| = 1 = |\zeta_2|$.

Next, $N_2$ is a free $B$-module so by (6.4)

$$H^*(D_8; N_2) \simeq H^*(\Sigma_2; H^0(B; N_2)) \simeq \mathbb{F}_2[\zeta]$$

with $|\zeta| = 1$.

Now, we have an extension

$$\Sigma_2 \rightarrow A \rightarrow \Sigma_2$$

so that

$$H^*(A; N_1) \simeq H^0(\Sigma_2; H^*(\Sigma_2; N_1)) \simeq \mathbb{F}_2[\eta], \quad |\eta| = 1,$$

since $N_1$ is a direct sum of two trivial $\Sigma_2$-modules with respect to the inner $\Sigma_2$-action and so proceed as above. Now, $N_2$ factors into $N_2' \oplus N_2''$ as $A$-modules where $N_2'$ is generated by $a \otimes b \otimes a \otimes b$ and $N_2''$ is generated by $a \otimes b \otimes b \otimes a$.

Thus

$$H^*(A; N_2) \simeq \mathbb{F}_2[\lambda'_1] \oplus \mathbb{F}_2[\lambda_2] \quad |\lambda_i| = 1 \quad i = 1, 2$$

by a computation as above.

**Proposition 7.11.** (1) Under the map $r(D_8, A): H^*(D_8; N) \rightarrow H^*(A; N)$

$$\zeta_1 \rightarrow \eta,$$

$$\zeta_2 \rightarrow \eta,$$

$$\zeta \rightarrow \lambda_1 \oplus \lambda_2.$$
(2) **Under the map** $c_{(2,3)} : H^*(\Delta; N) \to H^*(\Delta; N)$

$$\eta \mapsto \lambda_1,$$
$$\lambda_2 \mapsto \lambda_2.$$

(3) **Under the map** $t(\Delta, D_8) : H^*(\Delta; N) \to H^*(D_8; N)$

$$\eta' \mapsto 0,$$
$$\lambda'_1 \mapsto \xi'_r,$$
$$\lambda'_2 \mapsto \xi'_r$$

for all $r > 0$.

**Proof.** (1) Consider the diagram of extensions

$$\Sigma_2 \hookrightarrow \Delta \xrightarrow{\delta} \Sigma_2$$

where $\delta$ is the diagonal map. This induces

$$H^0(\Sigma_2; H^*(B; N_1)) \to H^0(\Sigma_2; H^*(\Sigma_2; N_1))$$

and

$$H^*(\Sigma_2; H^0(B; N_2)) \to H^*(\Sigma_2; H^0(\Sigma_2; N_2)).$$

These are the restriction maps

$$H^*(D_8; N_i) \to H^*(\Delta; N_i)$$

for $i = 1, 2$, by our above computations and Lemma 6.5. The first restriction is an easy computation. For the second restriction we have $H^0(B; N_2) \simeq \mathbb{F}_2$ and $H^0(\Sigma_2; N_2) \simeq \mathbb{F}_2 \oplus \mathbb{F}_2$ so that the induced map $\mathbb{F}_2 \to \mathbb{F}_2 \oplus \mathbb{F}_2$ is the diagonal map.

(2) This is an easy consequence of the fact that

$$c_{(2,3)} N_1 = N'_2,$$
$$c_{(2,3)} N''_2 = N''_2.$$

(3) First, $N_2$ is a free $B$-module so that $r(D_8, B)$ is trivial on $H^*(D_8; N_2)$ in positive degrees. Next, $N_1$ is a direct sum of two trivial $B$-modules thus

$$H^*(B; N_2) \simeq \mathbb{F}_2[\gamma_1, \gamma_2] \oplus \mathbb{F}_2[\phi_1, \phi_2]$$

where $|\gamma_i| = 1 = |\phi_i|$, $i = 1, 2$. From the diagram of extensions

$$\begin{array}{ccc}
B & \mapsto & B \\
\| & & \downarrow \\
B & \mapsto & D_8 \\
& & \Sigma_2
\end{array}$$
and Lemma 6.5, the restriction map \( r(D_8, B) \) on \( H^*(D_8; N_1) \) is equal to the inclusion
\[
H^0(\Sigma_2; H^*(B; N_1)) \to H^*(B; N_1).
\]
Thus
\[
r(D_8; B)_{\xi_1 \xi_2} = \gamma_1^{\xi_1} \gamma_2^{\xi_2} \oplus \phi_1^{\xi_1} \phi_2^{\xi_2}.
\]
We now pause to bring in the transfer

**Claim.**
\[
r(D_8, B)t(\Delta, D_8) = 0.
\]

**Proof.** By Proposition 6.6
\[
r(D_8, B)t(\Delta, D_8) = t(I, B)r(\Delta, I),
\]
where
\[
I = \Delta \cap B.
\]
Since \( I \) is a factor of \( B, r(B, I) \) is onto, but \( t(I, B)r(B, I) = 0 \) so that \( t(I, B) = 0 \).

From this claim and our computations, we conclude that
\[
t(\Delta, D_8)\lambda_i^e = c_i \xi_i^e,
\]
\( c_i \in \mathbb{F}_2, i = 1, 2 \). From Proposition 6.6, we have
\[
r(D_8, \Delta)t(\Delta, D_8) = 1 + c_{(1,2)}.
\]
Since
\[
(1, 2)N_1 = N_1,
\]
\[
(1, 2)N'_1 = N''_1,
\]
we get that under \( r(D_8, \Delta)t(\Delta, D_8) \)
\[
\lambda'_i \rightarrow \lambda'_i \oplus \lambda_2^e.
\]
So \( c_i = 1 \) for \( i = 1, 2 \). Finally, \( t(\Delta, D_8)\eta^e = 0 \) since \( \eta^e \) is in the image of \( r(D_8, \Delta) \).

Now, the relevance of the module \( N \) comes from

**Proposition 7.12.** (1) For \( 0 \leq i \leq m, 0 \leq j \leq n \),
\[
\Psi^{D_8}[\sigma_i(a), \sigma_j(b)] = \gamma_1^{m-i} \gamma_2^{n-j} \in H^*(D_8; N_1).
\]
(2) For $0 \leq i \leq m + n$

$$\psi^{D_8}\sigma_i[a, b] = \xi^{n+m-i} \in H^*(D_8; N_2).$$

**Proof.** These follow from Propositions 7.1–7.3. \]

Combining Corollary 6.3, (7.4), Propositions 7.5, 7.11, and 7.12 gives us our desired result.

**Proof of Proposition 5.7.** part (d): Again it is sufficient to prove the result for $V = K(m) \times K(n) \times K(p)$. Let $N$ be the $\Sigma_4$-submodule of $\pi_* V \otimes^4$ generated by $a \otimes a \otimes b \otimes d$. As such it is a summand of the $\Sigma_4$-module $\pi_* V \otimes^4$.

**Proposition 7.13.** For all $0 \leq i \leq m$

$$\psi^{D_8}[\sigma_i(a), [b, d]] \in H^m(D_8; N).$$

**Proof.** Again, this is a computation utilizing Propositions 7.1–7.3. \]

Now, since $N$ is a free $\Delta$-module, then by (7.4) and Proposition 7.5 the result follows from a computation utilizing Proposition 3.13 and Corollary 5.5.

**Proof of Proposition 5.7.** part (e): Let $V = K(m) \times K(n) \times K(p) \times K(q)$ and $N$ the $\Sigma_4$-submodule of $\pi_* V \otimes^4$ generated by $a \otimes b \otimes d \otimes e$.

**Proposition 7.14.**

$$\psi^{D_8}[[a, b], [d, e]] \in H^*(D_8; N).$$

**Proof.** Combine Propositions 7.2 and 7.3. \]

Now, $N$ is $\Sigma_4$-free so another computation using Proposition 3.13 and Corollary 5.5 gives us our result.

This completes the proof of Proposition 5.7.

**Proof of Proposition 5.10.** (a) It is sufficient to prove the result for $V = K(n)$ and $W = K(n)$. Suppose $n, m > 0$. Then

$$(N_V \otimes N_W)_* : \pi_* S_2 V \otimes S_2 W \rightarrow \pi_* S_2^2 V \otimes S_2 W$$

is injective. Thus, it suffices to compute $\pi_*^2$. By Theorem 6.1 and Lemma 5.9 our conclusion follows from $t(\Sigma_2, \Sigma_2 \times \Sigma_2) = 0$ since $r(\Sigma_2 \times \Sigma_2, \Sigma_2)$ is onto $H^*(\Sigma_2; \pi_*(V \otimes W) \otimes^2)$. Suppose $n = 0$. Define

$$i_1 : (S_2 V) \otimes W \rightarrow S_2(V \otimes W)$$

as the unique simplicial map such that

$$(xy) \otimes b \rightarrow (x \otimes b)(y \otimes b).$$
Also define
\[ i_2 : (S_2 V) \otimes W \to (S_2 V) \otimes (S_2 W) \]
as \( 1 \otimes i \) (see (2.6)). Then the diagram
\[
\begin{array}{ccc}
S_2(V \otimes W) & \xrightarrow{\phi''} & S_2 V \otimes S_2 W \\
\downarrow & & \downarrow \\
(S_2 V) \otimes W & & \\
\end{array}
\]
commutes. A computation gives the result. The case of \( m = 0 \) is the same.

(b) This is an easy computation using the diagram
\[
\begin{array}{ccc}
(V \otimes W)^{\otimes 2} & \xrightarrow{1 \otimes \tau \otimes 1} & S_2(V \otimes W) \\
\downarrow & & \downarrow \\
(V^{\otimes 2}) \otimes (W^{\otimes 2}) & \xrightarrow{\phi''} & S_2 V \otimes S_2 W \\
\end{array}
\]

Acknowledgements

This work was originally the author's Ph.D. Thesis done at M.I.T. He would like to thank his advisor, Haynes Miller, for many hours of productive discussion and sage advice. The author would also like to thank Bill Dwyer for pointing out a crucial observation, Paul Goerss for consultations on operations, Mike Hopkins for tutorials on group cohomology, and the referee for the useful suggestions toward a final rewrite. Finally he would like to thank Phil Hirschhorn and Judy Romvos for the tremendous LaTeX support they provided toward putting this final draft together.

References