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A manifold that does not contain a compact core

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Abstract

A core of a (noncompact) manifold is a submanifold with the property that the inclusion of the submanifold into the manifold is a homotopy equivalence. It is shown by example that a manifold may fail to contain a compact core even though the manifold has the homotopy type of a finite complex. © 1998 Elsevier Science B.V. All rights reserved.

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1. Introduction

In this note we construct examples of manifolds that do not contain compact cores. We begin with a definition that allows a precise statement of the problem considered.

Definition. Suppose M is a noncompact manifold and N is a codimension 0 submanifold (with boundary). We say that N is a *compact core* of M if N is compact and the inclusion map $N \hookrightarrow M$ is a homotopy equivalence.

The question we address is this: If M^n is an n-dimensional noncompact PL manifold that has the homotopy type of a finite k-dimensional complex K^k, then must M contain a compact core? The answer depends on the codimension, $n-k$. In case $n-k \geqslant 3$, it follows from Stallings' Embedding Theorem [13] that K can be embedded up to simple homotopy type in M. A regular neighborhood of the embedded copy of K is a PL submanifold that is a compact core of M. Thus compact cores always exist in

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codimensions greater than two. The main result of this paper asserts that such compact cores do not necessarily exist in codimension two. In addition, Ferry [1] has constructed examples which show that compact cores do not necessarily exist in codimension one either.

Theorem 1.1. *There exists an open subset W^n of S^n, $n \geqslant 6$, such that W^n has the homotopy type of $S^2 \times S^{n-4}$, but there is no compact submanifold N^n contained in W^n such that the inclusion $N^n \hookrightarrow W^n$ is a homotopy equivalence. There exists an open subset W^4 of S^4 such that W^4 has the homotopy type of S^2, but there is no compact submanifold N^4 contained in W^4 such that the inclusion $N^4 \hookrightarrow W^4$ is a homotopy equivalence.*

Theorem 1.1 implies that certain well known codimension three results do not generalize to codimension two. In particular, the Embedding Theorem of Stallings [13] mentioned above does not generalize to codimension two. Nor does the Browder–Casson–Haefliger–Sullivan–Wall Embedding Theorem [14, Corollary 11.3.4]. Stallings' Theorem asserts that any highly connected map $f : K \to M$ of a compact k-dimensional polyhedron into a (not necessarily compact) PL n-manifold, $n \geqslant k + 3$, is homotopic to a PL map $g : K \to M$ such that g is a simple-homotopy equivalence from K to a subcomplex K' of M. The BCHSW Theorem asserts that any highly connected map $f : N^k \to M^n$ from a compact, k-dimensional PL manifold without boundary into a (not necessarily compact) PL n-manifold, $n \geqslant k+3$, $n \geqslant 6$, is homotopic to a PL embedding. (The latter theorem is usually stated with a compact target, but it is also true in the noncompact case. If the target manifold M is not compact, a preliminary application of the Stallings' Theorem allows M to be replaced by a compact submanifold containing the image of f.)

In a forthcoming paper, [7], the main theorem of the present paper will be strengthened to show that there is no subcompactum that is a compact core of W^n. Specifically, there is no compact subset $X \subset W^n$ such that the inclusion $X \hookrightarrow W^n$ is a Čech homology equivalence. It follows that the two codimension three results mentioned above fail topologically in codimension two. In particular, there is a homotopy equivalence $f : S^2 \times S^{n-4} \to W^n$ that is not homotopic to any topological embedding, not even a wild one.

In dimension $n = 3$, the famous Core Theorem of Scott [11,12] asserts that any 3-manifold with finitely generated π_1 contains a compact core such that the inclusion is an equivalence on π_1. Our example shows that it is not possible to generalize this result to π_2 of 4-manifolds. Specifically, the 4-manifold W^4 mentioned in the theorem has the property that there is no compact submanifold such that the inclusion is an isomorphism on π_1 and π_2.

Dimension 4 is the minimal dimension in which the phenomena described in this paper can occur. If M^n has dimension $n \leqslant 3$ and has the homotopy type of a finite complex K^k with $k = n - 2$, then we can use general position to PL embed K in M and take a regular neighborhood of the embedded copy of K to form the core N. We suspect that compact cores do not necessarily exist in dimension 5, but we are unable to prove this.

The high dimensional proof does not work because taking a Cartesian product with S^1 increases the complexity of the first homology and so we are not able to compute the invariants we need.

The basic example used in this paper comes from [8]. Let W denote the open subset of S^4 described in [8] such that W has the homotopy type of S^2 but no nontrivial class in $H_2(W)$ can be represented by a piecewise linear 2-sphere. (The variant of the example of [8] that is an open subset of S^4 is described in [8, Section 5] and is denoted by W' there.) In Section 2 of this paper we give a new description of W in terms of the Kirby calculus of links [6]. In Section 3 we review the definition of the algebraic invariant we will use and then in Section 4 we prove that W has the properties spelled out in the theorem.

2. Construction of W

Let $\{L_i\}$ denote the sequence of 2-component links pictured in Fig. 1, below. Fig. 1 is the same as [8, Fig. 10], although it should be noted that some of the crossing of K_3' shown in [8, Fig. 10] are not correct. The links L_i are closely related to a family of links described by Milnor [9]. As in [8], the components of L_i are denoted K_i and m_i. Associated with each L_i there are two compact 4-manifolds: $M(L_i)$ denotes the 4-manifold obtained by attaching two 2-handles to B^4 along the components of L_i using zero framing; $W(L_i)$ denotes the 4-manifold obtained by attaching a 1-handle to B^4 using m_i as a guide and then adding a 2-handle along K_i using zero framing.

In terms of the Kirby calculus [6], $M(L_i)$ is the manifold whose diagram consists of L_i with 0's assigned to each component and $W(L_i)$ is the manifold whose diagram consists of L_i with a 0 assigned to K_i and a dot placed on m_i. It is clear that $M(L_i)$ has the homotopy type of $S^2 \vee S^2$ while $W(L_i)$ has the homotopy type of $S^2 \vee S^1$.

There is a simple relationship between $M(L_i)$ and $W(L_i)$. The 1-handle of $W(L_i)$ is actually constructed by taking a disk that m_i bounds, pushing the interior of the disk into B^4, and then removing a neighborhood of the disk from B^4. On the other hand, $M(L_i)$ is constructed by attaching an external 2-handle along the very same curve. As a result, we have a natural inclusion $W(L_i) \subset M(L_i)$. Furthermore, the disk removed to form the 1-handle on $W(L_i)$ along with the core of the 2-handle in $M(L_i)$ constitute a 2-sphere; hence

$$\overline{M(L_i) - W(L_i)} \cong S^2 \times B^2.$$

Since adding a 2-handle and removing a 2-handle have the same effect on the boundary, we see that $\partial W(L_i) \cong \partial M(L_i)$. Furthermore,

$$H_2\big(M(L_i), \partial M(L_i); \mathbb{Z}\big) \cong H^2\big(M(L_i); \mathbb{Z}\big) \cong \mathbb{Z} \oplus \mathbb{Z}.$$

The cocores of the 2-handles form a basis for $H_2(M(L_i), \partial M(L_i); \mathbb{Z})$. The fact that the components of L_i have homological linking number zero together with the fact that the handles are attached with zero framing implies that the boundary map

$$\partial : H_2\big(M(L_i), \partial M(L_i); \mathbb{Z}\big) \to H_1\big(\partial M(L_i); \mathbb{Z}\big)$$

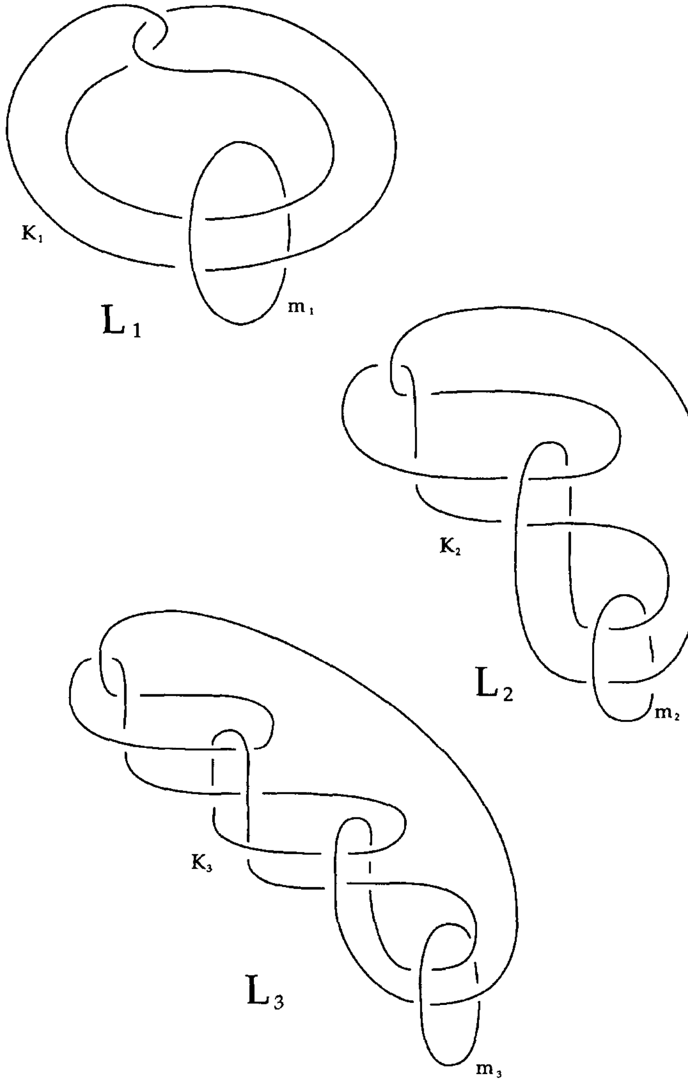

Fig. 1.

is an isomorphism. Thus the boundaries of the cocores (which equal the meridians of the two components of L_i) form a basis for $H_1(\partial M(L_i); \mathbb{Z}) \cong \mathbb{Z} \oplus \mathbb{Z}$.

Proposition 2.1. *The manifolds $W(L_i)$ can be constructed ambiently in S^4 so that*

(1) $W(L_i) \subset \operatorname{Int} W(L_{i+1}) \subset W(L_{i+1}) \subset S^4$,

(2) *the inclusion induced homomorphism $\pi_1(W(L_i)) \to \pi_1(W(L_{i+1}))$ is trivial, and*

(3) *the inclusion induced homomorphism $H_2(W(L_i)) \to H_2(W(L_{i+1}))$ is an isomorphism.*

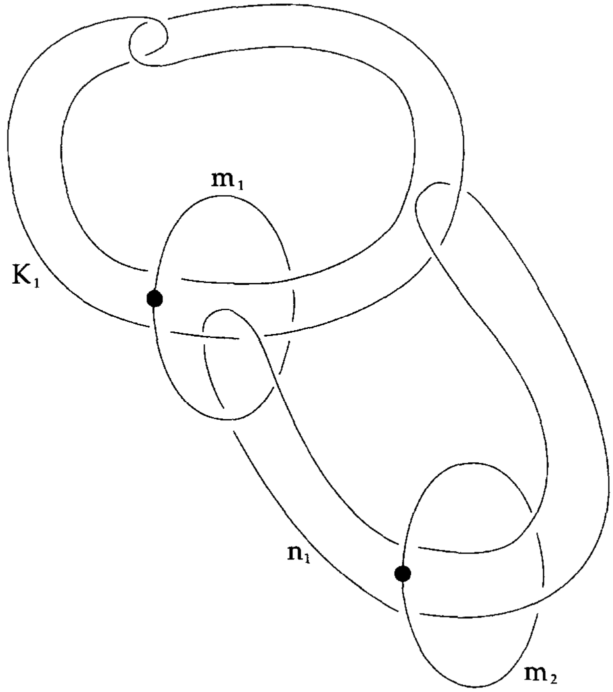

Fig. 2.

Proof. We proceed inductively to show that $W(L_i)$ can be constructed as in the statement of the proposition. Let us begin with $W(L_1)$. Notice that K_1 is an unknotted curve in S^3. Thus we can take $B^4 \subset S^4$, remove a neighborhood of a disk bounded by m_1 to form the 1-handle, and then realize the 2-handle by adding a neighborhood of a disk in $S^4 - \operatorname{Int} B^4$ that has K_1 as its boundary. This construction gives an embedding of $W(L_1)$ in S^4.

Next we explain how to attach a 1-handle and a 2-handle to $W(L_1)$ in S^4 to form $W(L_2)$. First thicken $W(L_1)$ by adding a small collar in S^4. Then add a 1-handle to $W(L_1)$ by removing a neighborhood of a disk whose boundary is the curve m_2 shown in Fig. 2. The neighborhood should be entirely contained in the collar added above so that the new manifold contains $W(L_1)$ in its interior.

Now attach a 2-handle along the curve n_1 that is also shown in Fig. 2. This can be done ambiently in S^4 because $K_1 \cup n_1$ is the unlink on ∂B^4: we find a smooth disk in $S^4 - \operatorname{Int} W(L_1)$ that has n_1 as its boundary and add a neighborhood of that disk to our manifold. The result is the addition of a 2-handle along n_1 with zero framing.

We claim that the manifold $W(L_1)$ + (1-handle) + (2-handle) is actually $W(L_2)$. In order to see this, we will cancel the $(1,2)$-handle pair represented by (m_1, n_1). Before doing that, we perform some handle slides to simplify the picture. First take the lower strand of K_1 as it passes through m_1 and slide it over the 1-handle represented by m_1.

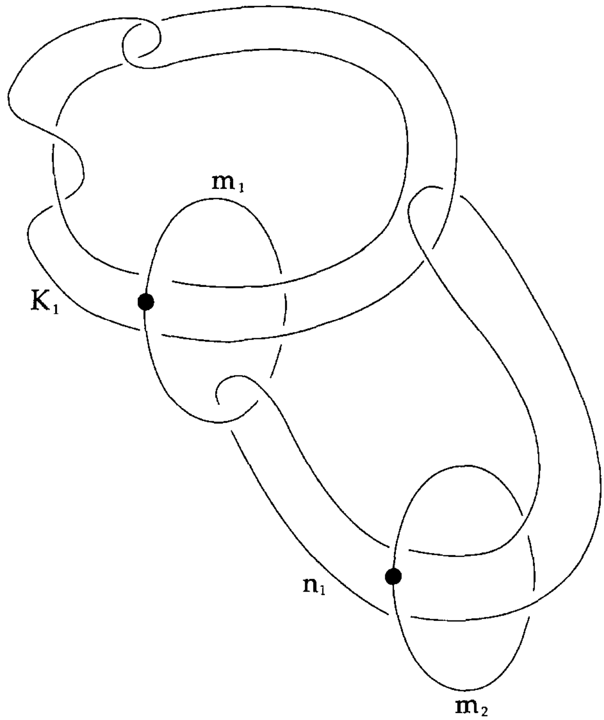

Fig. 3.

As shown in Fig. 3, this frees the strand of K_1 from n_1 at the expense of adding a full twist to K_1.

Next we slide the 2-handle represented by K_1 off the 1-handle represented by m_1. This is accomplished by twice sliding the 2-handle attached to K_1 over the new 2-handle attached along n_1. We do one handle slide for each of the two strands of K_1 that pass through m_1. This leaves the m_1 and n_1 linked to each other, but nothing else. Thus we can remove the the cancelling $(1, 2)$-handle pair that they represent from our picture. We use K_1' to denote K_1 in its new position. The result is shown in Fig. 4.

Now the link $K_1' \cup m_2$ is isotopic to L_2 so our new manifold actually is $W(L_2)$. All our constructions were done ambiently, hence we have $W(L_1) \subset W(L_2) \subset S^4$. Since K_2 is an unknotted curve, the construction can be continued inductively in order to prove the theorem. $\square$

We can now define the manifold W needed for the Main Theorem:

$$W = \bigcup_{i=1}^{\infty} W(L_i).$$

It follows from Proposition 2.1 that $W \subset S^4$ and that W has the homotopy type of S^2.

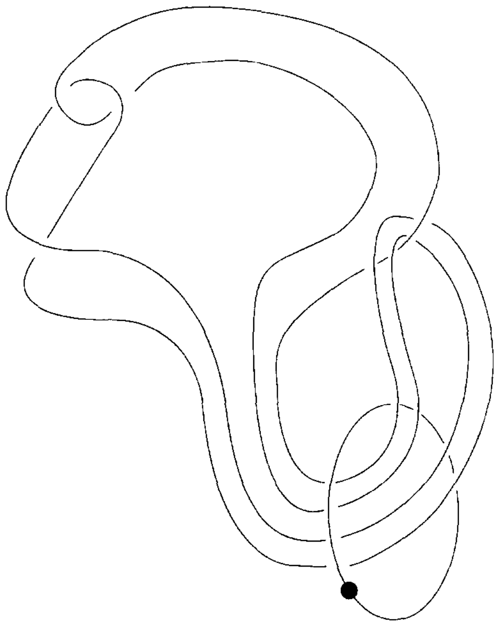

Fig. 4.

3. Definition of the invariant

In this section we define the algebraic invariant we will use. The definition is almost the same as that in [8]. The only difference is that we wish to define the invariant on a larger class of spaces.

First we establish some notation. We use $\mathbb{J}$ to denote the infinite cyclic group. We write $\mathbb{J}$ multiplicatively and use t to denote a generator. Thus $\mathbb{J} = \{t^k \mid k \in \mathbb{Z}\}$. Most homology groups will have rational coefficients and we use Λ to denote the group ring $\mathbb{Q}[\mathbb{J}]$. We think of the elements of Λ as Laurent polynomials in t with coefficients in $\mathbb{Q}$. It is important to remember that Λ is a principal ideal domain and that the only units in Λ are the monomials. We use the symbol $\doteq$ to indicate that two elements of Λ are equal up to multiplication by a unit. If C is a finitely generated module over Λ, then we can write

$$C \cong \frac{\Lambda}{(p_1(t))} \oplus \cdots \oplus \frac{\Lambda}{(p_\ell(t))}.$$

The ideal order C is defined to be the principal ideal in Λ generated by the product $p_1(t) \cdots p_\ell(t)$. This generator is well defined up to multiplication by a unit. The reader who wants to see a proof of this last statement or of any of the other assertions in this paragraph is referred to [2, Chapters III and IV].

Suppose X is a connected, separable metric space that is homotopy dominated by a (possibly infinite, but locally finite) CW complex. Let $\gamma : \pi_1(X) \to \mathbb{J}$ be a homomorphism. Up to homotopy, γ determines a map $f : X \to S^1$. (The existence of f is proved using the fact that X is dominated by a CW complex.) We define the *infinite cyclic cover* of X determined by γ to be the pullback

$$\widetilde{X} = \{(x,r) \in X \times \mathbb{R} \mid f(x) = e(r)\}$$

where $e:\mathbb{R} \to S^1$ is the standard covering map $e(r) = \exp(2\pi i r)$. Note that $\widetilde{X}$ is connected if and only if γ is an epimorphism. The covering map $p:\widetilde{X} \to X$ is defined by $p(x,r) = x$. The group $\mathbb{J}$ acts on $\widetilde{X}$ as a group of deck transformations according to the rule $t \cdot (x,r) = (x, r+1)$.

Definition of $A(X,\gamma;t)$. Suppose X is a connected, separable metric space that is homotopy dominated by a CW complex and that $\gamma:\pi_1(X) \to \mathbb{J}$ is a homomorphism. If $H_1(\widetilde{X};\mathbb{Q})$ is finitely generated over Λ, then we define the *Alexander Polynomial* $A(X,\gamma;t)$ of the pair (X,γ) to be a generator of order $T_1(\widetilde{X};\mathbb{Q})$, where $T_1(\widetilde{X};\mathbb{Q})$ is the Λ-torsion submodule of $H_1(\widetilde{X};\mathbb{Q})$. Note that $A(X,\gamma;t)$ is well-defined up to a product with units and that if $H_1(\widetilde{X};\mathbb{Q})$ is torsion free over Λ, then $A(X,\gamma;t) \doteq 1$. (This particular approach to the definition of Alexander Polynomial is due to Kawauchi [3,4].)

Definition of $k(X,\gamma)$. If X and γ are as above, then we can write $A(X,\gamma;t) = (t-1)^k B(t)$ with $B(1) \neq 0$ and $k \geqslant 0$. Define a nonnegative integer invariant $k(X,\gamma)$ by $k(X,\gamma) = k$. We refer to $k(X,\gamma)$ as the *Kawauchi Invariant* of (X,γ).

Alternative description of $k(X,\gamma)$. For any Λ-module C, the $(t-1)$*-primary torsion part of* C is defined to be

$$C_{(t-1)} = \{\, x \in C \mid (t-1)^m x = 0 \text{ for some } m \geqslant 0 \,\}.$$

It is not difficult to see that $k(X,\gamma)$ is equal to the $\mathbb{Q}$-dimension of the $(t-1)$-primary torsion part of $H_1(\widetilde{X};\mathbb{Q})$. In particular, $k(X,\gamma) = 0$ if and only if $H_1(\widetilde{X};\mathbb{Q})_{(t-1)} = \{0\}$.

We now state several lemmas that spell out the properties of $k(X,\gamma)$ that we will need later. These properties of $k(X,\gamma)$ are implicit in [8], but it seems worthwhile to state them formally as lemmas. If $\iota: X \hookrightarrow Y$ is the inclusion map and $\gamma:\pi_1(Y) \to \mathbb{J}$, we use $\gamma|X$ to denote the composite map $\pi_1(X) \xrightarrow{\iota_*} \pi_1(Y) \xrightarrow{\gamma} \mathbb{J}$.

Lemma 3.1. *Suppose X_1, X_2 and $X_1 \cup X_2$ are connected, separable metric spaces, each of which is dominated by a finite CW complex and suppose $\gamma:\pi_1(X_1 \cup X_2) \to \mathbb{J}$ is a homomorphism. If $X_1 \cap X_2 \neq \emptyset$ and $H_1(X_1 \cap X_2;\mathbb{Q}) = 0$, then $k(X_1 \cup X_2,\gamma) = k(X_1,\gamma|X_1) + k(X_2,\gamma|X_2)$.*

Proof. This is just a restatement of [8, Lemma 3.2]. □

Lemma 3.2. *If C is a finitely generated Λ-module and $t-1: C \to C$ is onto, then $C_{(t-1)} = \{0\}$.*

Proof. Note that $x \in C_{(t-1)}$ if and only if $(t-1)x \in C_{(t-1)}$. So the fact that $t-1: C \to C$ is onto implies that $(t-1)|C_{(t-1)}: C_{(t-1)} \to C_{(t-1)}$ is also onto. This means that for each $x \in C_{(t-1)}$ and for every nonnegative integer m there exists $y \in C_{(t-1)}$ such that $x = (t-1)^m y$. But $C_{(t-1)}$ is finitely generated over Λ (since C is), so there exists one m such that $(t-1)^m y = 0$ for every $y \in C_{(t-1)}$. It follows that $C_{(t-1)} = \{0\}$. □

Lemma 3.3. *Let X be a connected, separable metric space that is homotopy dominated by a CW complex and let $\gamma : \pi_1(X) \to \mathbb{J}$ be a homomorphism for which $H_1(\widetilde{X};\mathbb{Q})$ is finitely generated over Λ. If $H_1(X;\mathbb{Q}) \cong \mathbb{Q}$, and γ is an epimorphism, then $k(X,\gamma) = 0$.*

Proof. There is an exact sequence

$$H_1(\widetilde{X};\mathbb{Q}) \xrightarrow{t-1} H_1(\widetilde{X};\mathbb{Q}) \xrightarrow{\alpha} H_1(X;\mathbb{Q}) \xrightarrow{\beta} H_0(\widetilde{X};\mathbb{Q}) \xrightarrow{t-1} H_0(\widetilde{X};\mathbb{Q})$$

(see [10, p. 118]). Since γ is onto, $\widetilde{X}$ is connected. Thus each of the last three terms in the sequence is isomorphic to $\mathbb{Q}$, and the final homomorphism is trivial. It follows that β is onto. But every onto homomorphism of $\mathbb{Q}$ is an isomorphism, so β is one-to-one and $\alpha = 0$. Hence $t-1 : H_1(\widetilde{X};\mathbb{Q}) \to H_1(\widetilde{X};\mathbb{Q})$ is onto and Lemma 3.2 applies. □

Lemma 3.4. *Suppose $A \xrightarrow{\alpha} B \xrightarrow{\beta} C \xrightarrow{\gamma} D$ is an exact sequence of Λ-modules. If $t - 1 : A \to A$ is onto, B is finitely generated over Λ, and $D_{(t-1)} = \{0\}$, then $B_{(t-1)} \cong C_{(t-1)}$.*

Proof. It is clear that $\beta(B_{(t-1)}) \subset C_{(t-1)}$. So we show that $\beta|B_{(t-1)} : B_{(t-1)} \to C_{(t-1)}$ is an isomorphism. This follows from an elementary diagram chasing diagram involving the following diagram.

$$\begin{array}{ccccccc} A & \xrightarrow{\alpha} & B & \xrightarrow{\beta} & C & \xrightarrow{\gamma} & D \\ & & \cup\uparrow & & \uparrow\cup & & \\ & & B_{(t-1)} & \xrightarrow{\beta|B_{t-1}} & C_{(t-1)} & & \end{array}$$

□

Definition. The Λ-module C is said to be $(t-1)$*-divisible* if for each $x \in C$ there is a unique $y \in C$ such that $x = (t-1)y$. It is clear that if C is $(t-1)$-divisible, then $C_{(t-1)} = \{0\}$.

Our final result in this section is a version of [8, Lemma 3.3].

Lemma 3.5. *Suppose Y is a connected, separable metric space that is homotopy dominated by a CW complex and that $\gamma : \pi_1(Y) \to \mathbb{J}$ is a homomorphism. If $X \subset Y$, X is homotopy dominated by a finite complex, and $H_i(Y,X;\mathbb{Q}) = 0$ for $i \leqslant 2$, then $H_1(\widetilde{X};\mathbb{Q})_{(t-1)} \cong H_1(\widetilde{Y};\mathbb{Q})_{(t-1)}$.*

Proof. Let $p : \widetilde{Y} \to Y$ denote the infinite cyclic cover associated with γ. Then $\widetilde{X} = p^{-1}(X)$ is the infinite cyclic cover of X associated with $\gamma|X$. As in the proof of Lemma 3.3, there is an exact sequence

$$\cdots \to H_{i+1}(Y,X;\mathbb{Q}) \to H_i(\widetilde{Y},\widetilde{X};\mathbb{Q}) \xrightarrow{t-1} H_i(\widetilde{Y},\widetilde{X};\mathbb{Q}) \to H_i(Y,X;\mathbb{Q}) \to \cdots,$$

so $t-1 : H_i(\widetilde{Y}, \widetilde{X}; \mathbb{Q}) \to H_i(\widetilde{Y}, \widetilde{X}; \mathbb{Q})$ is onto for $i = 2$ and an isomorphism for $i = 1$. In particular, $H_1(\widetilde{Y}, \widetilde{X}; \mathbb{Q})$ is $(t-1)$-divisible. Since X is dominated by a finite complex, $H_1(\widetilde{X}; \mathbb{Q})$ is finitely generated over Λ. Thus Lemma 3.4 applies to the sequence

$$H_2(\widetilde{Y}, \widetilde{X}; \mathbb{Q}) \to H_1(\widetilde{X}; \mathbb{Q}) \to H_1(\widetilde{Y}; \mathbb{Q}) \to H_1(\widetilde{Y}, \widetilde{X}; \mathbb{Q})$$

and the proof is complete. □

4. Proof of the theorem

Throughout this section, W will denote the open subset of S^4 that was constructed in Section 2. We will apply the invariant of Section 3 to W in order to prove Theorem 1.1. We begin with a lemma that will allow us to use Lemma 3.3 in the proof of Theorem 1.1.

Lemma 4.1. *If N is a compact, connected submanifold of W such that $H_1(N; \mathbb{Z}) = 0$ and $H^2(N; \mathbb{Z}) \cong \mathbb{Z}$, then $H_1(\partial N; \mathbb{Z}) \cong \mathbb{Z}$.*

Proof. Since W is a subset of S^4, we may also consider N to be a subset of S^4. The Mayer–Vietoris sequence

$$H_2(\partial N; \mathbb{Z}) \to H_2(N; \mathbb{Z}) \oplus H_2(S^4 - \operatorname{Int} N; \mathbb{Z}) \to H_2(S^4; \mathbb{Z}) = 0$$

shows that the inclusion-induced homomorphism $\alpha : H_2(\partial N; \mathbb{Z}) \to H_2(N; \mathbb{Z})$ is onto. Therefore the boundary homomorphism $\partial : H_2(N, \partial N; \mathbb{Z}) \to H_1(\partial N; \mathbb{Z})$ in the following exact sequence is an isomorphism.

$$H_2(\partial N; \mathbb{Z}) \xrightarrow{\alpha} H_2(N; \mathbb{Z}) \to H_2(N, \partial N; \mathbb{Z}) \xrightarrow{\partial} H_1(\partial N; \mathbb{Z}) \to H_1(N; \mathbb{Z}) = 0.$$

Finally, Poincaré Duality shows that $H_2(N, \partial N; \mathbb{Z}) \cong H^2(N; \mathbb{Z})$ and so we have the desired conclusion. □

Since the manifold N in Lemma 4.1 is compact and $W = \bigcup_{i=1}^{\infty} \operatorname{Int} W(L_i)$, there must be an i such that $N \subset \operatorname{Int} W(L_i)$. In the proof of the main theorem, below, we will need to know that the generator of $H_1(\partial N; \mathbb{Z})$ can be identified with a particular loop on $\partial W(L_i)$. Recall that $W(L_i)$ is constructed by attaching one 2-handle and one 1-handle to B^4. Let b_i denote the belt sphere for the 2-handle. Then b_i is a loop on $\partial W(L_i)$ and b_i bounds a disk c_i in $W(L_i)$. (c_i is the cocore of the 2-handle.) Further, it is easy to see that (c_i, b_i) represents the generator of $H_2(W(L_i), \partial W(L_i); \mathbb{Z}) \cong H^2(W(L_i); \mathbb{Z}) \cong \mathbb{Z}$. The precise information we will need is contained in the following statement.

Addendum to Lemma 4.1. *If the restriction homomorphism*

$$H^2(W(L_i); \mathbb{Z}) \to H^2(N; \mathbb{Z})$$

is an isomorphism, then a generator of $H_1(\partial N; \mathbb{Z})$ is homologous to b_i in $W(L_i) - \operatorname{Int} N$.

Proof. Naturality of Poincaré Duality shows that the following diagram is commutative. (Coefficients in $\mathbb{Z}$ are understood.)

$$\begin{array}{ccccc} H_2(W(L_i),\partial W(L_i)) & \longrightarrow & H_2(W(L_i),W(L_i)-\operatorname{Int}N) & \xleftarrow[\cong]{\text{excision}} & H_2(N,\partial N) \\ \downarrow\cong & & & & \downarrow\cong \\ H^2(W(L_i)) & & & \longrightarrow & H^2(N) \end{array}$$

It follows that $(c_i \cap N, c_i \cap \partial N)$ represents a generator of $H_2(N,\partial N;\mathbb{Z})$. Since $\partial : H_2(N,\partial N;\mathbb{Z}) \to H_1(\partial N;\mathbb{Z})$ is an isomorpihsm, the set $c_i \cap \partial N$ represents a generator of $H_1(\partial N;\mathbb{Z})$. Furthermore, $c_i \cap \partial N$ is homologous to b_i via $c_i - \operatorname{Int} N$. □

Proof of Theorem 1.1 in case $n = 4$. We use the notation of Section 2. Suppose there exists a compact manifold $N \subset W$ such that $N \hookrightarrow W$ is a homotopy equivalence. We may assume that N has an external collar in W. (If not, first delete an open internal collar from N.) By compactness, there must exist an i such that $N \subset \operatorname{Int} W(L_i) \subset W(L_i) \subset M(L_i)$. As was pointed out in Section 2,

$$\overline{M(L_i) - W(L_i)} \cong S^2 \times B^2.$$

Let us define N_1 to be a neighborhood of $S^2 \times \{0\}$ in $\operatorname{Int} M(L_i) - N$ and define $N_2 = N$. We use $N_1 \natural N_2$ to denote the submanifold of $M(L)$ obtained by joining N_1 and N_2 with an arc and then thickening the arc to form a connected manifold. Notice that $\partial(N_1 \natural N_2)$ is homeomorphic to the connected sum of ∂N_1 and ∂N_2. Define $Y = \overline{M(L_i) - N_1 \natural N_2}$. The inclusion $N_1 \natural N_2 \hookrightarrow M(L_i)$ is a homotopy equivalence, so Y is a homology product. Since N has an external collar, Y is a topological manifold and the boundary of Y is bicollared in $M(L_i)$. It follows that Y is homotopy dominated by a close polyhedral neighborhood.

As noted in Section 2,

$$H_2(\partial M(L_i);\mathbb{Z}) \cong \mathbb{Z} \oplus \mathbb{Z},$$

and the generators are the belt spheres of the two 2-handles. Define $\gamma : \pi_1(\partial M(L_i)) \to \mathbb{J}$ by sending each of the generators of $H_1(\partial M(L_i);\mathbb{Z})$ to t. Since $\partial M(L_i) \hookrightarrow Y$ is a homology equivalence, γ may be extended to $\gamma : \pi_1(Y) \to \mathbb{J}$. Furthermore, γ defines $\gamma_1 : \pi_1(\partial N_1) \to \mathbb{J}$ and $\gamma_2 : \pi_1(\partial N_2) \to \mathbb{J}$ since $\partial(N_1 \natural N_2) \subset \partial Y$ is the connected sum of ∂N_1 and ∂N_2. The Addendum to Lemma 4.1 shows that each of γ_1 and γ_2 is an epimorphism. Thus Lemmas 4.1 and 3.3 show that $k(\partial N_1, \gamma_1) = 0 = k(\partial N_2, \gamma_2)$. By Lemma 3.1 we have $k(\partial(N_1 \natural N_2), \gamma|\partial(N_1 \natural N_2)) = 0$. Hence two applications of Lemma 3.5 give

$$k\big(\partial M(L_i), \gamma|\partial M(L_i)\big) = k(Y,\gamma) = k\big(\partial(N_1 \natural N_2), \gamma|\partial(N_1 \natural N_2)\big) = 0.$$

But $k(\partial M(L_i), \gamma|\partial M(L_i))$ is computed on pp. 213, 214 of [8]. The result is $k(\partial M(L_i), \gamma|\partial M(L_i)) = 2i$. This contradiction shows that our supposition that N exists must be false. □

Remarks. We have actually shown that there is no compact submanifold N of W such that N has the homotopy type of S^2 and the inclusion $N \hookrightarrow W$ is a homology equivalence. More information on how to compute k may be found in [5].

We now turn our attention to the higher dimensional cases of Theorem 1.1.

Lemma 4.2. *If N is a compact, connected submanifold of $W \times S^m$, $m \geqslant 2$, such that $H_1(N;\mathbb{Z}) = 0$ and $H^{m+2}(N;\mathbb{Z}) \cong \mathbb{Z}$, then $H_1(\partial N;\mathbb{Z}) \cong \mathbb{Z}$.*

Proof. The proof is nearly the same as that of Lemma 4.1. Note that $W \subset \mathbb{R}^4$ and $\mathbb{R}^4 \times S^m$ can be embedded in S^{4+m}, so we may think of N as a subset of S^{4+m}. The Mayer–Vietoris sequence

$$H_2(\partial N;\mathbb{Z}) \to H_2(N;\mathbb{Z}) \oplus H_2(S^{4+m} - \operatorname{Int} N;\mathbb{Z}) \to H_2(S^{4+m};\mathbb{Z}) = 0$$

shows that $\alpha : H_2(\partial N;\mathbb{Z}) \to H_2(N;\mathbb{Z})$ is onto. Therefore the boundary homomorphism $\partial : H_2(N,\partial N;\mathbb{Z}) \to H_1(\partial N;\mathbb{Z})$ in the following exact sequence is an isomorphism.

$$H_2(\partial N;\mathbb{Z}) \xrightarrow{\alpha} H_2(N;\mathbb{Z}) \to H_2(N,\partial N;\mathbb{Z}) \xrightarrow{\partial} H_1(\partial N;\mathbb{Z}) \to H_1(N;\mathbb{Z}) = 0.$$

Finally, Poincaré Duality shows that $H_2(N,\partial N;\mathbb{Z}) \cong H^{m+2}(N;\mathbb{Z})$ and so we have the desired conclusion. □

Just as in the previous case, we need to identify the generator of $H_1(\partial N;\mathbb{Z})$. By compactness there exists an i such that $N \subset \operatorname{Int} W(L_i) \times S^m$. Pick a point $* \in S^m$ and define b_i' to be the loop $b_i \times \{*\} \subset \partial W(L_i) \times S^m$ (where, as before, b_i is the boundary of the cocore of the 2-handle in $W(L_i)$). The proof of the following Addendum is exactly the same as that of the previous Addendum. (Simply replace c_i in the previous proof with $c_i \times \{*\}$.)

Addendum to Lemma 4.2. If the restriction homomorphism $H^{m+2}(W(L_i) \times S^m;\mathbb{Z}) \to H^{m+2}(N;\mathbb{Z})$ is an isomorphism, then the generator of $H_1(\partial N;\mathbb{Z})$ is homologous to b_i' in $(W(L_i) \times S^m) - \operatorname{Int} N$.

Proof of Theorem 1.1 in case $n \geqslant 6$. The proof follows the same outline as the proof in case $n = 4$. The main difference is the fact that the homology calculations are more delicate.

Let $W^n = W \times S^{n-4}$. Then W^n has the homotopy type of $S^2 \times S^{n-4}$. Further, W is a subset of $\mathbb{R}^4$, so $W^n \subset \mathbb{R}^4 \times S^{n-4}$. But $\mathbb{R}^4 \times S^{n-4}$ can be embedded in S^n, so we may assume $W^n \subset S^n$. Suppose W^n contains a compact core N; then N has the homotopy type of $S^2 \times S^{n-4}$. By compactness there must exist an i such that $N \subset \operatorname{Int} W(L_i) \times S^{n-4}$. Let N' denote a regular neighborhood of the core of $(M(L_i) - W(L_i)) \times S^{n-4} \cong S^2 \times S^{n-4} \times B^2$ such that N' is disjoint from $N \subset W(L_i) \times S^{n-4} \subset M(L_i) \times S^{n-4}$. Form a connected submanifold C of $M(L_i) \times S^{n-4}$ by connecting N and N' with an arc and then thickening the arc a little. In order to simplify the notation in the rest of the proof, let us use M to denote $M(L_i) \times S^{n-4}$ and let $Y = M - \operatorname{Int} C$. Notice that ∂Y is the disjoint union of ∂M and ∂C.

We claim that $H_j(Y,\partial M;\mathbb{Z}) = 0$ and $H_j(Y,\partial C;\mathbb{Z}) = 0$ for $j \leqslant 2$. In order to see this, note that M has the homotopy type of $(S^2 \vee S^2) \times S^{n-4}$ and that C has the homotopy type of $(S^2 \times S^{n-4}) \vee (S^2 \times S^{n-4})$. (Here '$\vee$' denotes the 'wedge product' or one point union.) Hence $H_j(C;\mathbb{Z}) = H_j(M;\mathbb{Z}) = 0$ for $j \leqslant 1$ and $H_2(C;\mathbb{Z}) \to H_2(M;\mathbb{Z})$ is onto. (The Künneth Theorem may be used to compute $H_2(M;\mathbb{Z})$ and $H_2(C;\mathbb{Z})$. In case $n \geqslant 7$, $H_2(C;\mathbb{Z}) \cong \mathbb{Z}\oplus\mathbb{Z} \cong H_2(M;\mathbb{Z})$. In case $n = 6$, $H_2(C;\mathbb{Z}) \cong \mathbb{Z}\oplus\mathbb{Z}\oplus\mathbb{Z}\oplus\mathbb{Z}$ and $H_2(M;\mathbb{Z}) \cong \mathbb{Z}\oplus\mathbb{Z}\oplus\mathbb{Z}$. Thus $H_2(C;\mathbb{Z}) \to H_2(M;\mathbb{Z})$ is an isomorphism in case $n \geqslant 7$ and is onto in case $n = 6$.) The exact sequence of the pair (M, C) shows that $H_j(M, C;\mathbb{Z}) = 0$ for $j \leqslant 2$ and so excision gives us that $H_j(Y, \partial C;\mathbb{Z}) = 0$ for $j \leqslant 2$. On the other hand, Alexander Duality shows that

$$H_j(Y, \partial M;\mathbb{Z}) \cong H^{n-j}(M, C;\mathbb{Z}).$$

Now $H^{n-j}(M;\mathbb{Z}) = 0 = H^{n-j}(C;\mathbb{Z})$ for $j = 0, 1, 3$ and $H^{n-2}(M;\mathbb{Z}) = \mathbb{Z}\oplus\mathbb{Z} = H^{n-2}(C;\mathbb{Z})$, so $H^{n-j}(M, C;\mathbb{Z}) = 0$ for $j \leqslant 2$.

We now define $\gamma : \pi_1(Y) \to \mathbb{J}$. Since $\partial M = (\partial M(L_i)) \times S^{n-4}$, we may define $\gamma|\partial M$ to be the composition of the homomorphism induced by the projection $\partial M \to \partial M(L_i)$ and the homomorphism $\pi_1(\partial M(L_i)) \to \mathbb{J}$ that was used in the proof of the case $n = 4$, above. By the preceding paragraph, $\gamma|\partial M$ extends uniquely to $\gamma : \pi_1(Y) \to \mathbb{J}$. Also using the preceding paragraph along with Lemma 3.5 we see that

$$k(\partial M, \gamma|\partial M) = k(Y, \gamma) = k(\partial C, \gamma|\partial C).$$

But $k(\partial M, \gamma|\partial M) = 2i$ and $k(\partial C, \gamma|\partial C) = 0$; this contradiction shows that no compact core N can exist. The fact that $k(\partial C, \gamma|\partial C) = 0$ is demonstated exactly as in the proof of the $n = 4$ case, using Lemma 4.2 in place of Lemma 4.1. The fact that $k(\partial M, \gamma|\partial M) = 2i$ follows from the observation that the infinite cyclic cover of ∂M associated with γ is simply the Cartesian product of the cover of $\partial M(L_i)$ with S^{n-4}. Since $n \geqslant 6$, the S^{n-4} factor does not contribute anything to the first homology. □

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