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AN APPROXIMATION THEOREM IN SHAPE THEORY

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In this paper it is shown that if X is a compactum in the interior of a PL manifold M and if U is a neighborhood of X in M, then there is a compactum X' in U such that X and X' have the same relative shape in U and the embedding dimension of X' equals the fundamental dimension of X. Whenever the dimension of M is not equal to three, the relative shape equivalence from X' to X can be realized by an infinite isotopy of M.

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Introduction

Suppose X is a compactum embedded in the interior of a piecewise linear (PL) n-manifold M^n. In [8] and [9] a condition on the embedding of X into M^n (called the inessential loops condition [8]) was studied. It was shown that for compacta X which satisfy this condition there is a strong connection between the shape of X and the geometry of neighborhoods of X in M^n. The question naturally arises whether it is reasonable to expect an embedding to satisfy the inessential loops condition. In this note we answer that question by showing that, in the shape category, every embedding of a compactum can be approximated by one for which the embedding dimension is equal to the fundamental dimension (definitions below). In particular, if the fundamental dimension of X is less than or equal to $(n-3)$, then X can be approximated in the shape category by compacta which satisfy the inessential loops condition (by general position).

Theorem. *Suppose X is a compactum in the interior of the PL n-manifold M^n. Then for every neighborhood U of X in M there exists a compactum $X' \subset U$ such that*
(a) $\operatorname{dem}(X') = \operatorname{Fd}(X)$, *and*
(b) *X and X' have the same relative shape in U.*

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Remark. The approximation theorem together with the addendum below and the main theorem of [9] give a theory in the shape category which is exactly analogous to the demension theory of Štan'ko [7] in the topological category.

Addendum. *If* $(n, \mathrm{Fd}(X)) \neq (3, 1)$, *then there exists a map* $H: M \times [0, \infty) \to M$ *such that*

(c) *the maps* $\{H_i | U\}_{i=0}^{\infty}$ *form a relative fundamental sequence from* X' *to* X,

(d) *each* $H_t: M \to M$ *is a* PL *homeomorphism, and*

(e) *for each neighborhood* W *of* X *there exists an integer* i *such that* $\mathrm{supp}(H_t \circ H_i^{-1}) \subset W$ *for all* $t \geq i$.

The following consequence of the approximation theorem was pointed out to the author by Steve Ferry. The result was previously known to R. Geoghegan and J. Hollingsworth. The analogous statement in codimension one is not true [3].

Corollary. *If* $X \subset \mathbb{R}^n$ *and* $\mathrm{Sh}(X) = \mathrm{Sh}(K)$ *where* X *is a compactum and* K *is a finite polyhedron of dimension* $k \leq n-3$, *then there exists a finite polyhedron* $K' \subset \mathbb{R}^n$ *such that* $\dim K' = k$ *and* K' *has the simple-homotopy type of* K.

Definitions. For our purposes, the following definition of *demension* (=dimension of embedding) is most covenient. If X is a compactum in the interior of a PL manifold M, then $\mathrm{dem}\, X \leq k$ if for every $\varepsilon > 0$ there exists a compact k-dimensional polyhderon K and a regular neighborhood N of K such that $X \subset \mathrm{int}\, N$ and N is an ε-mapping cylinder neighborhood of K.

If X is a compact metric space (=compactum), then the *fundamental dimension* of X is defined by $\mathrm{Fd}(X) = \min\{\dim Y \mid Y \text{ is a compactum with } \mathrm{Sh}(X) \leq \mathrm{Sh}(Y)\}$.

The reader is referred to [2, pp. 267 and 268] for the definition of relative shape. The idea is that two subsets X and X' of U have the same relative shape in U if there are fundamental sequences from X to X' and from X' to X which map U to U which are mutual inverses and are homotopic to the inclusions in U. The relative shape depends not only on X and X', but also on U and the embedding of X and X' in U. See [1] for all other definitions related to shape theory.

Proof of the Theorem. Let $k = Fd(X)$.

Case 1: $k \geq n-2$, $(n, k) \neq (3, 1)$. In this case, $X = \bigcap_{i=1}^{\infty} N_i$ where each $N_i \subset \mathrm{int}\, N_{i-1} \subset N_{i-1} \subset U$ and N_i is a regular neighborhood of a compact polyhedron K_i with $\dim K_i \leq k$ [9, Theorem 1.4]. There exists a PL homeomorphism $h_1: U \to U$ such that h_1 is isotopic to the identity, $h_1 | K_1 = \mathrm{id}$ and $h_1(N_1)$ is a $\frac{1}{2}$-mapping cylinder neighborhood of K_1. Similarly there exists a PL homeomorphism h_2 such that $h_2 \circ h_1(N_2)$ is a $\frac{1}{4}$-mapping cylinder neighborhood of $h_1(K_2)$ and inductively there exists an h_i such that $h_i \circ \cdots \circ h_1(N_i)$ is a $(1/2^i)$-mapping cylinder neighborhood of $h_{i-1} \circ \cdots \circ h_1(K_i)$. Let $X' = \bigcap_{i=1}^{\infty} h_i \circ \cdots \circ h_1(N_i)$. $\square$

Case 2: $(n, k) = (3, 1)$. By [9, Proposition 2.1] there exists a sequence of neighborhoods $\{U_i\}_{i=0}^{\infty}$ such that $U_0 = U$, $U_i \subset U_{i-1}$, and the inclusion map $U_i \hookrightarrow U_{i-1}$ is homotopic in U_{i-1} to a map $\beta_i : U_i \to K_i$ where K_i is a 1-dimensional polyhderon in U_{i-1}. By general position we may assume that $\beta_i | K_{i+1}$ is an embedding; in fact, that β_i is a homeomorphism on a small regular neighborhood N_{i+1} of K_{i+1}. Let $X' = \bigcap_{i=1}^{\infty} \beta_1 \circ \beta_2 \circ \cdots \circ \beta_i(N_{i+1})$. $\square$

Case 3: $k \leq n-3$. By [9, Proposition 2.1] again, there exists a sequence $\{U_i\}_{i=0}^{\infty}$ of neighborhoods of X and finite k-dimensional polyhdedra $K_i \subset U_i$ such that $U_0 = U$, $U_{i+1} \subset U_i$ and for each $i \leq 0$ there exists a homotopy $f_i : U_{i+1} \times [0, 1] \to U_i$ such that $f_i(x, 0) = x$ and $f_i(\cdot, 1) \in K_i$ for every $x \in U_{i+1}$. For each $i \geq 0$, we amalgamate $K_{i\cdot k}, K_{i\cdot k+1}, \ldots, K_{i\cdot k+(k-1)}$ to form a k-dimensional polyhedron L_i as follows:

(1) Let $L_{i,1} = K_{(i+1)\cdot k-1}$.

(2) Inductively let

$$L_{i,j} = K_{(i+1)\cdot k-j} \cup f_{(i+1)\cdot k-j}(L_{i,j-1}^{(k-1)} \times [0, 1]) \cup L_{i,j-1} \quad \text{for } 2 \leq j \leq k.$$

(3) Define L_i to be $L_{i,k}$.

Here the superscript $(k-1)$ denotes the $(k-1)$-dimensional skeleton. The maps f_i can be adjusted slightly so that each $L_{i,j}$ is a polyhderon. Let $U_i' = U_{i\cdot k}$ and define $\beta_i : U_{i+1}' \to L_i$ by $\beta_i(x) = f_{i\cdot k+(k-1)}(x, 1)$ for each $i \geq 0$.

Engulfing Lemma. *For every polyhedron $P \subset U_{i+1}'$ with $\dim P \leq k$ and for every regular neighborhood N of L_i, there exists a* PL *isotopy h_t of U_i' with compact support such that $h_0 = \mathrm{id}$ and $h_1(N) \supset P$. Furthermore, $h_1^{-1} | P$ is homotopic to $\beta_i | P$ in N.*

We will assume the Engulfing Lemma for now and finish the proof of Case 3.

Let N_i be a regular neighborhood of L_i for each i. By the lemma (with $P = L_{i+1}$), there exists a homeomorphism $g_i : U_i' \to U_i'$ such that $g_i(L_{i+1}) \subset N_i$ and $g_i | L_{i+1}$ is homotopic to $\beta_i | L_{i+1}$ in N_i. By making N_{i+1} smaller if necessary, we can arrange that $g_i(N_{i+1}) \subset N_i$ and that $g_i | N_{i+1}$ is homotopic to $\beta_i | N_{i+1}$ in N_i. Define $G_i : N_{i+1} \to N_0$ by $G_i = g_0 \circ g_1 \circ \cdots \circ g_i$ and let $X' = \bigcap_{i=1}^{\infty} G_i(N_{i+1})$. The regular neighborhoods N_i can be chosen inductively, as the lemma is applied, so that dem $X' = k$. We must check that X and X' have the same relative shape in $U = U_0$.

Now $X = \bigcap_{i=0}^{\infty} U_i'$ and $X' = \bigcap_{i=0}^{\infty} G_i(N_{i+1})$. We define maps $f_i : U_{i+1}' \to G_{i-1}(N_i)$ and $f_i' : G_i(N_{i+1}) \to U_{i+1}'$ by $f_i = G_{i-1} \circ \beta_i$ and $f_i' = G_i^{-1}$. Since all the maps g_i, g_i^{-1} and β_i are homotopic to the inclusions in U, we have that f_i and f_i' are homotopic to the inclusions in U. Furthermore, $f_i \circ f_i' : G_i(N_{i+1}) \to G_{i-1}(N_i)$ is equal to

$$g_0 \circ g_1 \circ \cdots \circ g_{i-1} \circ \beta_i \circ g_i^{-1} \circ g_{i-1}^{-1} \circ \cdots \circ g_1^{-1} \circ g_0^{-1}$$

and thus is homotopic to the inclusion in $G_{i-1}(N_i)$ by construction of g_i. The map $f'_{i-1} \circ f_i: U'_{i+1} \to U'_i$ is just β_i and consequently is homotopic to the inclusion in U'_i. Since $\beta_{i+1}: U'_{i+2} \to N_{i+1} \hookrightarrow U'_{i+1}$ is homotopic to the inclusion and $\beta_i | N_{i+1}$ is homotopic to $g_i | N_{i+1}$, we have that f_{i+1} is homotopic to $f_i | U'_{i+2}$ in $G_{i-1}(N_i)$. Also f'_{i+1} is homotopic to $f'_i | G_{i+1}(N_{i+2})$ in U'_{i+1} because g_i^{-1} is homotopic to the identity.

The relative fundamental sequences needed to complete the proof are now constructed by carefully extending f_i and f'_i in the usual way (using the homotopy extension property). □

Proof of the Addendum. In the proof above, $f'_i = G_i^{-1} = g_i^{-1} \circ g_{i-1}^{-1} \circ \cdots \circ g_0^{-1}$ and each g_i^{-1} is isotopic to the identiy in U'_i via an isotopy with compact support. Hence we can define a map $H: M \times [0, \infty) \to M$ by letting $H | M \times [i, i+1]$ be g_i^{-1} composed with the isotopy between the identity and g_{i+1}^{-1}. The map H has the property that each $H_t: M \to M$ defined by $H_t(x) = H(x, t)$ is a PL homeomorphism. □

Proof of the Lemma. We use all the notation from the proof of Case 3 and also let $L_{i,0} = \emptyset$. We actually prove the following inductive statement.

If $p \leq k$, a finite p-dimensional polyhedron $P \subset U_{i \cdot k+p}$ can be engulfed with N keeping $L_{i,k-p}$ fixed.

If $p = 0$ or if $p = k = 1$, the result is easy. Suppose $p = 1 < k$. We must construct a homotopy of P into N which keeps $P \cap L_{i,k-1}$ fixed. There is a homotopy of P into N which keeps $P \cap L_{i,k-1}$ in N: first push $P \cap L_{i,k-1}$ along $L_{i,k-1}$ until it is near $I^{(k}_{i}{}^{)}_{1}$ and then use the homotopy $f_{i \cdot k}$ to pull P into a neighborhood of $K_{i \cdot k}$. By squeezing out the fibers of that homotopy which lies over $P \cap L_{i,k-1}$, we get a homotopy $g: P \times [0, 1] \to U'_i$ such that $g_0 = \mathrm{id}$, $g_1(P) \subset N$ and $g(x, t) = x$ for every $x \in P \cap L_{i,k-1}$. Put g in general position on $(P - L_{i,k-1}) \times [0, 1]$, keeping $g((P \cap L_{i,k-1}) \times [0, 1])$ fixed. Then g will embed $(P - L_{i,k-1}) \times [0, 1]$ in $U'_i - L_{i,k-1}$. Push N out along that embedded homotopy to engulf P.

Now suppose that $p < k$ and that the inductive statement above is true for polyhedra of dimension $p' < p$. First construct a homotopy

$$g: P \times [0, 1] \to U_{i \cdot k+(p+1)}$$

such that $g_0 = \mathrm{id}$, $g | (P \cap L_{i,k-p}) \times [0, 1] = \mathrm{id}$, and $g(P + \{1\}) \subset L_{i,k-p+1}$ (just as above). Put $g | (P - L_{i,k-p}) \times [0, 1]$ into general position, keeping

$$g((P \cap L_{i,k-p}) \times [0, 1] \cup g(P \times \{1\}))$$

fixed and let

$$S = S(g | (P - L_{i,k-p}) \times [0, 1]) \cup g^{-1}(g((P - L_{i,k-p}) \times [0, L)) \cap L_{i,k-p}).$$

Then

$$\dim S \leq (p+1) + k - n \leq (p+1) + (n-3) - n = p-2.$$

Let Σ denote the shadow of S. We have $\dim \Sigma \leq p-1$. By induction, there exists an isotopy h_t' of U_i' such that $h_1'(N) \supset g(\Sigma)$ and $h_t'|L_{i,k-p+1} = \mathrm{id}$. Thus $g(P \times \{1\} \cup \Sigma) \subset h_1'(N)$. But

$$g(P\times[0,1]) \searrow g(P\times\{1\} \cup \Sigma \cup (P \cap L_{i,k-p}) \times [0,1])$$

and

$$g(P\times\{1\} \cup \Sigma \cup (P \cap L_{k,k-p}) \times [0,1]) \supset g(P \times [0,1]) \cap L_{i,k-p},$$

and so we can find the isotopy needed to finish the proof of this case by simply following the inverse of the collapse.

Finally, suppose that $p = k \leq n-3$. Let $g: P\times[0,1] \to U_{(i+1)\cdot k-1}$ be a homotopy such that $g_0 = \mathrm{id}$ and $g_1(P) = \beta_i(P) \subset K_{(i+1)\cdot k-1}$. Put g in general position keeping $g(P\times\{1\})$ fixed. (Recall that $L_{i,k-k} = \emptyset$.) By Zeeman's Piping Lemma [10, Lemma 48], g can be adjusted so that there is a polyhedron $J \subset P \times [0,1]$ such that:

(1) $S(g) \subset J$,

(2) $\dim J \leq k-1$,

(3) $\dim(J \cap (P^{(k-1)} \times [0,1])) \leq k-2$, and

(4) $P\times[0,1] \searrow J \cup P^{(k-1)} \times [0,1] \cup P \times \{1\}$.

Let Σ denote the shadow of $J \cap (P^{(k-1)} \times [0,1])$. Note that

$$P\times[0,1] \searrow P^{(k-1)}\times[0,1] \cup P\times\{1\} \cup J \searrow P\times\{1\} \cup \Sigma \cup J$$

by property (4) and the definition of Σ. Properties (2) and (3) imply that $\dim(\Sigma \cup J) \leq k-1$. By induction, there exists an isotopy h_t' of U_i' such that $h_0' = \mathrm{id}$, $h_1'(N) \supset \Sigma \cup J$ and $h_t'|L_{i,1} = \mathrm{id}$. Thus $g(P\times\{1\} \cup \Sigma \cup J) \subset h_1'(N)$ and we can engulf the rest of $g(P\times[0,1])$ by following the inverse of the collapse of $P\times[0,1]$ to $P\times\{1\} \cup \Sigma \cup J$.

The proof of the inductive statement is now complete. To finish the proof of the lemma, we merely observe that in the last case (in which $P \subset U_{i+1}'$) the isotopy h_t constructed has the properties that $h_1(N) \supset g(P\times[0,1])$ and $h_t|g_1(P) = \mathrm{id}$. Since $g_1|P = \beta_i|P$, we have that $h_1^{-1}|P$ is homotopic to $h_1^{-1}\beta_i||P = \beta_i|P$ in N. □

Proof of the Corollary. If $n \leq 5$, the corollary is obvious because K can be embedded in $\mathbb{R}^n$ by general position. Suppose $X \subset \mathbb{R}^n$ where $\mathrm{Sh}(X) = \mathrm{Sh}(K)$ and K is a connected finite k-dimensional complex, $k \leq n-3$, $n \geq 6$. By the theorem, there exists $X' \subset \mathbb{R}^n$ such that $\mathrm{Sh}(X') = \mathrm{Sh}(K)$ and dem $X' \leq k$. By [4, Theorem 2.5], X' has an I-regular neighborhood U in $\mathbb{R}^n$. The neighborhood U has one end ε and $\pi_1(\varepsilon) = \pi_1(U - X') = \pi_1(U)$. By [5, Proposition 6.11], the obstruction to finding a collar for ε is equal to the obstruction to U having finite type. But U is a complex and so U has the homotopy type of K which means that the obstruction vanishes. Let M be a compact PL manifold equal to U minus an open collar of ε. Then $X' \hookrightarrow M$ is a shape equivalence. By the main theorem of [6], there exists a finite polyhedron $K' \subset M$ such that $\dim K' = k$ and K and K' have the same simple-homotopy type. □

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