

## EMBEDDINGS OF LOCALLY CONNECTED COMPACTA

BY

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**ABSTRACT.** Let  $X$  be a  $k$ -dimensional compactum and  $f: X \rightarrow M^n$  a map into a piecewise linear  $n$ -manifold,  $n \geq k + 3$ . The main result of this paper asserts that if  $X$  is locally  $(2k - n)$ -connected and  $f$  is  $(2k - n + 1)$ -connected, then  $f$  is homotopic to a CE equivalence. In particular, every  $k$ -dimensional,  $r$ -connected, locally  $r$ -connected compactum is CE equivalent to a compact subset of  $\mathbf{R}^{2k-r}$  as long as  $r \leq k - 3$ .

**Introduction.** Let  $X$  be a  $k$ -dimensional compactum. In this paper we study the problem of finding an embedding of  $X$  into Euclidean  $n$ -space  $\mathbf{R}^n$ . Specifically, we wish to investigate conditions under which we can embed  $X$  in  $\mathbf{R}^n$  when  $n$  is less than the classical dimension  $2k + 1$ .

In case  $X$  is a manifold, it has long been known that such improvements are possible. Whitney [Wh] showed that every smooth  $k$ -dimensional manifold embeds in  $\mathbf{R}^{2k}$ . His techniques were later generalized to codimension three embedding theorems for piecewise linear (PL) manifolds by Irwin [Ir] and Hudson [Hd] who showed that if the manifold is  $r$ -connected, then it is possible to PL embed it in  $\mathbf{R}^{2k-r}$ .

The situation for polyhedra is different. In order to accomplish the embedding into a lower-dimensional space it is necessary to identify certain contractible sets to points and this changes the homeomorphism type of the polyhedron. Thus the map produced is not a topological embedding, but rather a simple homotopy equivalence. So the appropriate theory for polyhedra is a theory of embedding up to simple homotopy type and this theory was worked out by Stallings [St]. It follows from his main theorem that every  $k$ -dimensional,  $r$ -connected polyhedron is simple homotopy equivalent to a subpolyhedron of  $\mathbf{R}^{2k-r}$ .

The purpose of this paper is to prove a theorem like that of Stallings, but for more general compacta rather than polyhedra. In moving from polyhedra to compacta it is necessary to add an additional hypothesis: We must assume that the polyhedra are locally  $r$ -connected as well as globally  $r$ -connected. In addition, we must use an appropriate generalization of the (strictly polyhedral) concept of simple homotopy equivalence. We substitute CE equivalence. This is the natural thing to do because

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every CE map of polyhedra induces a simple homotopy equivalence [Ch] and a simple homotopy equivalence is made up of a sequence of maps, each of which shrinks out cell-like sets of a particular kind.

Before stating our main theorem we give a definition of CE equivalence.

**DEFINITION.** A compact space  $A$  is said to be a *cell-like set* if, for some embedding of  $A$  into an absolute neighborhood retract (ANR),  $A$  has the property that  $A$  is contractible in every neighborhood of itself. A map  $f: X \rightarrow Y$  of compacta is said to be a *CE map* if  $f$  is onto and  $f^{-1}(y)$  is a cell-like subset of  $X$  for every  $y \in Y$ . We say that  $f$  is a *CE equivalence* if there exists a compactum  $Z$  and CE maps  $g: Z \rightarrow X$  and  $h: Z \rightarrow Y$  such that  $h = fg$ .

**THEOREM 1.** Suppose  $X$  is a locally  $r$ -connected,  $k$ -dimensional compactum and  $M^n$  is a PL  $n$ -manifold,  $k \leq n - 3$ ,  $r \geq 2k - n$ . If  $f: X \rightarrow M$  is an  $(r + 1)$ -connected map, then  $f$  is homotopic to a CE equivalence; i.e., there exist  $k$ -dimensional compacta  $Y$  and  $Z$  such that  $Y \subset M^n$  and CE maps  $Z \rightarrow X$  and  $Z \rightarrow Y$  such that

$$\begin{array}{ccc} X & \xrightarrow{f} & M \\ \uparrow & & \cup \\ Z & \rightarrow & Y \end{array}$$

commutes up to homotopy. Furthermore, if  $r \leq k - 2$  then  $Y$  is locally  $r$ -connected.

**COROLLARY.** If  $X$  is a  $k$ -dimensional,  $r$ -connected, locally  $r$ -connected compactum,  $r \leq k - 3$ , then  $X$  is CE equivalent to a compact,  $k$ -dimensional, locally  $r$ -connected subset of  $\mathbf{R}^{2k-r}$ .

We note that Theorem 1 has precisely the same connectivity conditions as are found in Stallings' theorem, except that we assume local connectivity as well as global connectivity. The local connectivity is automatic in the case of polyhedra and the example in [D-H] shows that it is a necessary hypothesis in the case of compacta.

There have been other generalizations of [St] to compacta. One such generalization involves the use of shape equivalence instead of CE equivalence. The main results of that type are contained in [H-I]. It is interesting to note that the hypotheses used in [H-I] are equivalent (by [Fe1]) to the assumption that  $X$  has the shape of an  $LC^r$  compactum for precisely the same value of  $r$  as is used here.

In addition, Husch [Hs] has proved a metastable range theorem similar to ours but using a condition which is stronger than the  $LC^r$  hypothesis. He proves that if  $X$  can be expanded as the inverse limit of an inverse sequence of  $k$ -dimensional polyhedra with  $UV^r$  bonding maps,  $r \geq 2k - n$ ,  $k \leq n - 3$ , and  $k < (2/3)(n - 1)$ , then any  $(r + 1)$ -connected map  $f: X \rightarrow M^n$  is homotopic to a CE equivalence. Ferry [Fe1] has shown that if  $X$  can be written as an inverse limit of an inverse sequence with  $UV^r$  bonding maps, then  $X$  is  $LC^r$  (while the converse is not true). Thus Husch's theorem follows from Theorem 1. But the proof of Theorem 1 given in this paper is based on Husch's construction.

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**1. Definitions.** A space  $X$  is said to be  $r$ -connected if every map of the  $m$ -sphere  $S^m$ ,  $m \leq r$ , into  $X$  extends to a map of the  $(m + 1)$ -ball  $B^{m+1}$  into  $X$ . We say that  $X$  is *locally  $r$ -connected* (abbreviated  $LC^r$ ) if for every  $x \in X$  and for every neighborhood  $U$  of  $x$  there exists a neighborhood  $V$  of  $x$  such that any map of  $S^m$  into  $V$  extends to a map of  $B^{m+1}$  into  $U$ ,  $m \leq r$ . A map  $f: X \rightarrow Y$  is said to be an  $r$ -connected map if  $\pi_i(M(f), X) = 0$  for  $i \leq r$ . Here  $M(f)$  denotes the mapping cylinder of  $f$ .

Let  $\varepsilon$  be a positive number. A homotopy  $f: X \times [0, 1] \rightarrow Y$  is said to be an  $\varepsilon$ -homotopy if  $\text{diam}(f(\{x\} \times [0, 1])) < \varepsilon$  for every  $x \in X$ . A collapse of one polyhedron to another induces a strong deformation retraction in a natural way. The collapse is said to be an  $\varepsilon$ -collapse if the induced homotopy is an  $\varepsilon$ -homotopy. A regular neighborhood  $N$  of a polyhedron  $K$  is said to be an  $\varepsilon$ -regular neighborhood if  $N$  collapses to  $K$  via an  $\varepsilon$ -collapse.

All polyhedra considered in this paper are compact. If  $L$  is a polyhedron equipped with a triangulation, then  $L^{(r)}$  denotes the  $r$ -skeleton.

Inverse limits are useful in this paper. If  $\{X_i\}$  is a sequence of spaces and  $f_i: X_{i+1} \rightarrow X_i$ , we consider  $\varprojlim \{X_i, f_i\}$  to be the subset of the product space  $\prod X_i$  consisting of all sequences  $(x_i)$  such that  $f_i(x_{i+1}) = x_i$ .

It is also convenient to use shape theory occasionally to prove that certain sets are cell-like. We use only the most elementary facts which can be found in any book on the subject (see [D-S], for example).

Finally, we mention taming theory. We follow [Ed and Št] and say that a  $k$ -dimensional compactum  $X$  in a PL manifold is *tame* if for every  $\varepsilon > 0$  there exists a neighborhood  $N$  of  $X$  such that  $N$  is an  $\varepsilon$ -regular neighborhood of a  $k$ -dimensional polyhedron.

**2. Neighborhoods.** Let  $X$  denote a  $k$ -dimensional, compact,  $LC^r$  metric space which is a subset of the interior of a PL  $n$ -manifold  $M$ . In this section we will build neighborhoods of  $X$  in  $M$  which reflect the local connectivity of  $X$ . Specifically, this means that the neighborhoods should contain small homotopies pushing polyhedra of appropriate dimension into  $X$ .

**DEFINITION.** Let  $\varepsilon$  be a small positive number. We will call a neighborhood  $N$  of  $X$  an  $(\varepsilon, r)$ -deformation neighborhood if for every  $(r + 1)$ -dimensional polyhedron  $L$  and every map  $f: L \rightarrow N$  there exists an  $\varepsilon$ -homotopy  $f_t: L \rightarrow N$  such that  $f_0 = f$  and  $f_1(L) \subset X$ . We say that  $N$  is a *strong*  $(\varepsilon, r)$ -deformation neighborhood if  $N$  has the additional property that if  $L_0$  is a subpolyhedron of  $L$  such that  $f(L_0) \subset X$ , then we can require that  $f_t|_{L_0} = f|_{L_0}$  for every  $t$ .

In this section we will prove that, under suitable hypotheses,  $LC^r$  compacta have arbitrarily small strong  $(\varepsilon, r)$ -deformation neighborhoods. The converse is also true: if  $X$  has strong  $(\varepsilon, r)$ -deformation neighborhoods for arbitrarily small  $\varepsilon$ , then  $X$  is

$LC^r$ . This is because a map of  $S^m$  into a small subset of  $X$  extends to a map of  $B^{m+1}$  into a small subset of  $M$ . If  $m \leq r$ , a strong  $(\varepsilon, r)$ -deformation neighborhood can then be used to push the image of  $B^{m+1}$  into  $X$ .

It seems simplest to begin by proving the existence of (weak)  $(\varepsilon, r)$ -deformation neighborhoods. The proof of the following lemma was communicated to the author by M. Bestvina.

**LEMMA 2.1.** *Suppose  $X$  is a  $k$ -dimensional compactum which is tamely embedded in the PL  $n$ -manifold  $M$ . If  $X$  is  $LC^r$ ,  $r \leq k - 2$ ,  $k + r + 3 \leq n$ , then for every  $\varepsilon > 0$  there is an  $(\varepsilon, r)$ -deformation neighborhood  $N$  of  $X$  in  $M$  such that  $N$  is an  $\varepsilon$ -regular neighborhood of a  $k$ -dimensional polyhedron  $K$ .*

**PROOF.** The proof is by induction on  $r$  and begins with the case  $r = -1$  (in which case there is something to prove even though there is no restriction on  $X$ ).

Choose  $\delta > 0$  such that for any point  $x \in N_\delta(X)$  there exists a path of diameter  $< \varepsilon/5$  from  $x$  to a point of  $X$ . Let  $N_1$  be a neighborhood of  $X$  such that  $N_1$  is an  $(\varepsilon/5)$ -regular neighborhood of a  $k$ -dimensional polyhedron  $K_1$  and  $N_1 \subset N_\delta(X)$ . Take a triangulation of  $K_1$  having mesh less than  $\varepsilon/5$ .

For each vertex  $v$  of  $K_1$ , choose an  $(\varepsilon/5)$ -arc  $B_v$  from  $v$  to a point of  $X$ . We may assume that  $B_v \cap K_1 = \{v\}$ . (If the other end of  $B_v$  lies in  $K_1 \cap X$ , just move  $K_1$  slightly.) Keep the endpoints of  $B_v$  fixed and push the rest of  $B_v$  out of  $N_1$  using the natural  $(\varepsilon/5)$ -product structure on  $N_1 - K_1 \cong \partial N_1 \times [0, 1]$ . The new arc,  $C_v$ , has diameter  $< 3\varepsilon/5$ . Now add a small  $(\varepsilon/5)$ -regular neighborhood of each arc  $C_v$  to  $N_1$ . We take  $N$  to be  $N_1$  plus the neighborhoods of the arcs and take  $K$  to be  $K_1$  plus the arcs  $C_v$  plus the shadows of  $C_v \cap \partial N_1$  under the collapse of  $N_1$  to  $K_1$ .

Pick a point  $p \in N$ . We must show that there is an  $\varepsilon$ -homotopy of  $p$  in  $N$  which pushes  $p$  into  $X$ . First homotope  $p$  into  $K$  with an  $(\varepsilon/5)$ -homotopy. In case this pushes  $p$  into  $K_1$ , continue to push  $p$  along  $K_1$  to the nearest vertex. The other case is that the first homotopy pushes  $p$  into one of the arcs  $C_v$ . In either case we get  $p$  into one of the  $C_v$  with a  $(2\varepsilon/5)$ -homotopy. The homotopy is completed by moving along  $C_v$  into  $X$ .

Now suppose that  $r \geq 0$  and that  $X$  has  $(\varepsilon, r - 1)$ -deformation neighborhoods for arbitrarily small  $\varepsilon$ .

Let  $\varepsilon > 0$  be given. Choose  $\delta_1 > 0$  such that any  $\delta_1$ -map of  $S^{r+1}$  into the  $\delta_1$ -neighborhood of  $X$  extends to an  $(\varepsilon/8)$ -map of  $B^{r+2}$  into  $M$ . (The existence of  $\delta_1$  follows from the fact that  $M$  is  $LC^{r+1}$ .) Next we use the fact that  $X$  is  $LC^r$  to choose  $\delta > 0$  such that  $\delta < \delta_1/2$  and if  $L$  is any  $(r + 1)$ -complex and  $f: L^{(r)} \rightarrow X$  such that  $\text{diam}[f(\partial\Delta)] < \delta$  for every  $(r + 1)$ -simplex  $\Delta$  in  $L$ , then  $f$  extends to a map  $f': L \rightarrow X$  such that  $\text{diam}[f'(\Delta)] < \delta_1/2$  for every simplex  $\Delta$  in  $L$ .

Now apply the inductive hypothesis to get a  $(\delta/3, r - 1)$ -deformation neighborhood  $N_1$  of  $X$  such that  $N_1$  collapses to a  $k$ -dimensional polyhedron  $K_1$  via an  $(\varepsilon/8)$ -collapse. Triangulate  $K_1$  with a triangulation of mesh less than  $\delta/3$  and let  $K_1^{(r+1)}$  denote the  $(r + 1)$ -skeleton of  $K_1$ .

We claim that there exists an  $(\varepsilon/8)$ -homotopy of  $K_1^{(r+1)}$  into  $X$  which keeps  $K_1^{(r)}$  in  $N_1$ . By the choice of  $N_1$ , there exists a  $(\delta/3)$ -homotopy  $h_r: K_1^{(r)} \rightarrow N_1$  such that  $h_0 = \text{inclusion}$  and  $h_r(K_1^{(r)}) \subset X$ . Extend  $h_0$  via the inclusion to all of  $K_1^{(r+1)}$ . For

each  $(r + 1)$ -simplex  $\Delta \subset K_1^{(r+1)}$ ,  $h_1(\partial\Delta) \subset X$  and  $\text{diam}[h_1(\partial\Delta)] < \delta$ . So the choice of  $\delta$  allows us to extend  $h_1$  to  $K_1^{(r+1)}$  in such a way that  $h_1(K_1^{(r+1)}) \subseteq X$  and  $\text{diam}[h_1(\Delta)] < \delta_1/2$  for each  $\Delta \subseteq K_1^{(r+1)}$ . Then for each  $(r + 1)$ -simplex  $\Delta \subseteq K_1^{(r+1)}$ ,  $h_0(\Delta) \cup h_1(\Delta) \cup \{h_t(\partial\Delta) | 0 \leq t \leq 1\}$  has diameter less than  $\delta_1$ . The choice of  $\delta_1$  now allows us to extend  $h_t$  to the  $(\varepsilon/8)$ -homotopy whose existence was claimed above.

We now have a map  $h: K_1^{(r+1)} \times [0, 1] \rightarrow M$  such that  $h(x, 0) = x$  for all  $x \in K_1^{(r+1)}$ ,  $h(K_1^{(r+1)} \times \{1\}) \subseteq X$ ,  $h(K_1^{(r)} \times [0, 1]) \subseteq N_1$ , and  $\text{diam}[h(\Delta \times [0, 1])] < \varepsilon/8$  for every  $(r + 1)$ -simplex  $\Delta \subset K_1^{(r+1)}$ . For some small positive number  $\gamma$ , make  $h|_{K_1^{(r+1)} \times [0, 1 - \gamma]}$  PL and put it in general position with respect to  $K_1$ . The condition  $k + r + 3 \leq n$  then implies that  $h(K_1^{(r+1)} \times [0, 1 - \gamma])$  intersects  $K_1$  only along  $h(K_1^{(r+1)} \times \{0\}) = K_1^{(r+1)}$ . We may also assume that  $\gamma$  and the PL approximation were chosen carefully enough so that

$$h(K_1^{(r+1)} \times \{0, 1 - \gamma\}) \cup h(K_1^{(r)} \times [0, 1 - \gamma]) \subset N_1.$$

Push the rest of  $h(K_1^{(r+1)} \times [0, 1 - \gamma])$  out of  $N_1$ , keeping  $h(K_1^{(r+1)} \times \{0, 1 - \gamma\}) \cup h(K_1^{(r)} \times [0, 1 - \gamma])$  fixed. Then  $h(K_1^{(r+1)} \times [0, 1 - \gamma])$  consists of the part lying in  $N_1$  together with some  $(r + 2)$ -cells poking out of  $N_1$ . Each of these  $(r + 2)$ -cells has diameter less than  $3\varepsilon/8$ . We take  $N$  to be  $N_1$  plus  $(\varepsilon/8)$ -regular neighborhoods of the  $(r + 2)$ -cells and take  $K$  to be  $K_1$  plus the  $(r + 2)$ -cells plus shadows of the  $(r + 2)$ -cells in the product structure of  $N_1 - K_1$ .

We must now show that  $N$  satisfies the conclusion of the lemma. Let  $L$  be an  $(r + 1)$ -dimensional polyhedron and  $f: L \rightarrow N$ . We construct the homotopy of  $f(L)$  into  $X$  in several stages. First, homotope  $f(L)$  out of the  $(r + 2)$ -handles and into  $N_1$ . Next, push  $f(L)$  down the product structure of  $N_1 - K_1$  into  $K_1$ . Then push along  $K_1$  into  $K_1^{(r+1)}$ . From there on, use the homotopy  $h_t$ . The second and third of those homotopies are  $(\varepsilon/8)$ -homotopies and the first and last are  $(3\varepsilon/8)$ -homotopies, so the entire homotopy is an  $\varepsilon$ -homotopy.  $\square$

**LEMMA 2.2.** *Suppose  $X$  is a compact  $LC^0$  subset of the PL manifold  $M$ . Then for every  $\varepsilon > 0$  there exists a  $\delta > 0$  such that every  $(\delta, 0)$ -deformation neighborhood of  $X$  is a strong  $(\varepsilon, 0)$ -deformation neighborhood of  $X$ .*

**PROOF.** Let  $\varepsilon > 0$  be given. Choose  $\delta > 0$  such that  $3\delta \leq \varepsilon/6$  and such that if  $x, y \in X$  and  $d(x, y) < 3\delta$ , then  $x$  and  $y$  can be joined by an arc in  $X$  of diameter less than  $\varepsilon/6$ .

Now suppose that  $N$  is a  $(\delta, 0)$ -deformation neighborhood of  $X$  in  $M$ . Let  $L$  be a 1-dimensional polyhedron,  $L_0$  a subpolyhedron of  $L$  and  $f: L \rightarrow N$  a map such that  $f(L_0) \subseteq X$ . Choose a triangulation of  $L$  of such small mesh that  $\text{diam}[f(\Delta)] < \delta$  for every simplex  $\Delta$  in  $L$ . For each vertex  $v$  of  $L - L_0$ , there is a  $\delta$ -homotopy of  $f(v)$  into  $X$ . Extend those homotopies via the identity to  $f(L_0)$  and then to a  $\delta$ -homotopy of all of  $f(L)$  in  $N$ .

Now if  $\Delta$  is a 1-simplex in  $L$ , we have  $\delta$ -homotoped  $f|_\Delta$  to a map  $f': \Delta \rightarrow N$  such that  $f'(\partial\Delta) \subseteq X$  and  $\text{diam}[f'(\Delta)] < 3\delta \leq \varepsilon/6$ . Thus the proof of the lemma will be complete if we can show that  $f'$  is  $(5\varepsilon/6)$ -homotopic, rel  $\partial\Delta$ , to a map of  $\Delta$  into  $X$ . To do so we use a base point trick just like that in [L-V, Lemma 1].

By the choice of  $\delta$ , we can extend  $f'|_{\partial\Delta}$  to a map  $f'': \Delta \rightarrow X$  such that  $\text{diam}[f''(\Delta)] < \varepsilon/6$ . Let  $\Omega$  be the loop made up of  $f'(\Delta)$  together with  $f''(\Delta)$  with reverse orientation. We want to show that  $\Omega$  is homotopic, rel base point, to a loop  $\Omega'$  in  $X$ . If so, then  $f'(\Delta)$  is homotopic (rel base point) to the path  $f''(\Delta)$  followed by  $\Omega'$  which lies entirely in  $X$  and so we are finished.

Take a triangulation of  $N$  of mesh less than  $\delta$  and let  $K$  denote the 1-skeleton of the triangulation. By hypothesis, there exists a  $\delta$ -homotopy  $h_1: K \rightarrow N$  such that  $h_0 = \text{inclusion}$  and  $h_1(K) \subset X$ . By moving  $f''$  slightly (if necessary) we may assume that there is a point  $y \in K$  such that  $h_1(y) \in f''(\Delta)$ . This may increase the size of  $f''(\Delta)$ , but only by  $\varepsilon/6$ . We will first use  $y$  as a base point. Let  $a$  be the path from  $y$  to  $h_1(y)$  traced out by the homotopy  $h_1|_{\{y\}}$ . Notice that there is a base-point-fixing  $2\delta$ -homotopy of the loop  $a\Omega a^{-1}$  to a loop of the form  $a\Omega' a^{-1}$  where  $\Omega'$  is a loop in  $X$ . (This homotopy consists of a push into the 1-skeleton  $K$ , followed by the homotopy  $h_1$ .) But now using  $h_1(y)$  as a base point, we see that  $\Omega \simeq a^{-1}a\Omega a^{-1} \simeq a^{-1}a\Omega' a^{-1} \simeq \Omega'$  where " $\simeq$ " denotes "is homotopic to" and all the homotopies are rel the base point  $h_1(y)$ . To complete the proof we need only compute the size of the homotopies involved. The entire image of the homotopy has

$$\begin{aligned} \text{diameter} &\leq \text{diam}[\Omega] + \text{diam}[a] + 4\delta + \text{diam}[a] \\ &\leq (\text{diam}[f'(\Delta)] + \text{diam}[f''(\Delta)] + \varepsilon/6) + \delta + 4\delta + \delta \\ &\leq 3\delta + \varepsilon/6 + \varepsilon/6 + 6\delta < 5\varepsilon/6. \quad \square \end{aligned}$$

The following is the main result of this section.

**PROPOSITION 2.3.** *Suppose  $X$  is a  $k$ -dimensional compactum which is tamely embedded in the PL  $n$ -manifold  $M$ . If  $X$  is  $LC^r$ ,  $r \leq k - 2$ , and  $k + r + 3 \leq n$ , then for every  $\varepsilon > 0$  there is a strong  $(\varepsilon, r)$ -deformation neighborhood  $N$  of  $X$  in  $M$  such that  $N$  is an  $\varepsilon$ -regular neighborhood of a  $k$ -dimensional polyhedron  $K$ .*

**REMARK.** The hypothesis  $r + k + 3 \leq n$  could be weakened considerably. But doing so would require extra work and we do not have any need for a stronger result in this paper.

**PROOF.** The proof is by induction on  $r$ . The case  $r = -1$  has already been done because there is no difference between an  $(\varepsilon, -1)$ -deformation neighborhood and a strong  $(\varepsilon, -1)$ -deformation neighborhood. The case  $r = 0$  is covered by Lemmas 2.1 and 2.2. So we may assume inductively that  $r \geq 1$  and that for every  $\varepsilon > 0$  there is a  $\delta > 0$  such that every  $(\delta, r - 1)$ -deformation neighborhood of  $X$  is a strong  $(\varepsilon, r - 1)$ -deformation neighborhood of  $X$ .

Let  $\varepsilon > 0$  be given. Use the fact that  $X$  is  $LC^r$  to choose  $\delta_1 > 0$  such that any map  $g: S^r \rightarrow X$  with  $\text{diam}[g(S^r)] < 3\delta_1$  extends to a map  $G: B^{r+1} \rightarrow X$  such that  $\text{diam}[G(B^{r+1})] < \varepsilon/5$ . By the inductive hypothesis there exists  $\delta > 0$  such that any  $(\delta, r - 1)$ -deformation neighborhood of  $X$  is a strong  $(\delta_1, r - 1)$ -deformation neighborhood. We may also assume that  $\delta \leq \delta_1$  and  $3\delta_1 \leq \varepsilon/5$ .

By Lemma 2.1 there exists a neighborhood  $N$  of  $X$  such that  $N$  is a  $(\delta, r)$ -deformation neighborhood and such that  $N$  is a  $\delta$ -regular neighborhood of a  $k$ -dimensional polyhedron  $K$ . We will show that such an  $N$  is a strong  $(\varepsilon, r)$ -deformation neighborhood.

Let  $L$  be an  $(r + 1)$ -dimensional polyhedron,  $L_0$  a subpolyhedron of  $L$  and  $f: L \rightarrow N$  a map such that  $f(L_0) \subset X$ . We must produce an  $\varepsilon$ -homotopy  $f_t: L \rightarrow N$  such that  $f_0 = f$ ,  $f_1(L) \subset X$ , and  $f_t|_{L_0} = f|_{L_0}$  for every  $t$ . We may assume that  $L_0$  is  $r$ -dimensional, because otherwise we could restrict our attention to  $\text{Clos}(L - L_0)$ , construct such a homotopy of  $\text{Clos}(L - L_0)$  which keeps  $L_0 \cap \text{Clos}(L - L_0)$  fixed and then extend it via  $f$  to all of  $L = \text{Clos}(L - L_0) \cup L_0$ .

Pick a triangulation for  $L$  such that if  $\Delta$  is a simplex in  $L$ , then  $\text{diam}[f(\Delta)] < \delta$  and  $L_0$  is a subcomplex of this triangulation. By the choice of  $N$ , there is a  $\delta_1$ -homotopy  $h_t: L \rightarrow N$  such that  $h_0 = f$ ,  $h_t|_{L_0} = f|_{L_0}$  for each  $t$ , and  $h_1(L^{(r)}) \subset X$ . For each  $(r + 1)$ -simplex  $\Delta$  of  $L$ , we have that  $h_1(\partial\Delta) \subset X$  and  $\text{diam}[h_1(\Delta)] < 3\delta_1$ . Just as in the proof of Lemma 2.2, we will be finished if we can show that there is a  $(4\varepsilon/5)$ -homotopy of  $h_1(\Delta)$  into  $X$  which keeps  $h_1(\partial\Delta)$  fixed.

So consider one such  $\Delta$ . By the choice of  $\delta_1$ , there is a map  $h': \Delta \rightarrow X$  which extends  $h_1|_{\partial\Delta}$  and such that  $\text{diam}[h'(\Delta)] < \varepsilon/5$ . Consider the singular  $(r + 1)$ -sphere  $S = h_1(\Delta) \cup h'(\Delta)$ . There is a  $\delta$ -homotopy of  $S$  to a singular  $(r + 1)$ -sphere  $S' \subset X$ . Pick a base point  $x \in S \cap X$ . The path  $p$  followed by  $x$  during the homotopy from  $S$  to  $S'$  is a  $\delta$ -path beginning and ending on  $X$ . By induction, there exists a  $\delta_1$ -homotopy of  $p$  into  $X$  which keeps the ends of  $p$  fixed. Let  $p'$  be the new path in  $X$ . Notice that  $\text{diam}[p'] \leq \delta + 2\delta_1$  and that  $S$  is homotopic (rel base point) to the singular  $(r + 1)$ -sphere  $S'' \subset X$  which is  $S'$  acted on by the path  $p'$ . Therefore  $h_1|_{\Delta}$  is homotopic, rel  $\partial\Delta$ , to a map of  $\Delta$  onto  $S'' \cup h'(\Delta)$ .

The size of that homotopy is no greater than  $\text{diam}[S] + 2\delta + 2\delta_1$ . But  $\text{diam}[S] \leq 3\delta_1 + \varepsilon/5$ , so the size of the homotopy is no more than  $3\delta_1 + \varepsilon/5 + 2\delta + 2\delta_1 < 4\varepsilon/5$ .  $\square$

Proposition 2.3 seems like the simplest, most natural way to state the properties of the neighborhoods we have constructed. But in the proof of Theorem 1 we will actually need the following, more technical, statement.

**PROPOSITION 2.4.** *Let  $X$  be an  $LC^r$  compactum in the PL  $n$ -manifold  $M$  and let  $\varepsilon$  be a positive number. Suppose  $N_1$  is a strong  $(\varepsilon, r)$ -deformation neighborhood of  $X$ ,  $\rho: N_1 \rightarrow K_1$  is an  $\varepsilon$ -retraction onto a  $k$ -dimensional spine  $K_1$ ,  $N_0 \subseteq N_1$  is a second  $(\varepsilon, r)$ -deformation neighborhood of  $X$ ,  $L$  is an  $(r + 1)$ -dimensional polyhedron and  $L_0$  is a subpolyhedron of  $L$ . Then if  $f_0: L_0 \rightarrow N_0$  is a map such that  $\rho f_0$  extends to a map  $f: L \rightarrow K_1$ , then there is an extension  $f': L \rightarrow N_0$  of  $f_0$  such that  $f$  and  $\rho f'$  are  $5\varepsilon$ -homotopic, rel  $L_0$ , in  $K_1$ .*

**PROOF.** We will show that there is a  $3\varepsilon$ -homotopy  $g_t: L \rightarrow N_1$  such that  $g_0 = f$ ,  $g_1(L) \subset N_0$ ,  $g_t|_{L_0} = f_0$  and  $\rho(g_t(x)) = f(x)$  for every  $x \in L_0$  and for every  $t$ . Once such a homotopy has been constructed, we simply take  $f' = g_1$  and notice that the homotopy  $\rho g_t$  is a  $5\varepsilon$ -homotopy from  $\rho g_0 = f$  to  $\rho g_1 = \rho f'$ .

To construct  $g_t$  we proceed as follows. Pick a barycentric subdivision of  $L$  having very small mesh. Let  $A$  denote the simplicial neighborhood of  $L_0$  in this triangulation,  $C = \text{Clos}(L - A)$  and  $B = A \cap C$ . Since  $N_0$  is an  $(\varepsilon, r)$ -deformation neighborhood, there exists an  $\varepsilon$ -homotopy of  $f(L_0)$  in  $N_0$  which pushes  $f(L_0)$  into  $X$ . By

spreading that homotopy out over the neighborhood  $A$ , we construct a map  $F: L \times [0, 1] \rightarrow N_1$  such that  $F(x, 0) = f(x)$  for all  $x \in L$ ,  $F(x, 1) = f_0(x)$  for all  $x \in L_0$ ,  $F(x, 1) \in N_0$  for all  $x \in A$ ,  $F(x, 1) \in X$  for all  $x \in B$ ,  $\rho(F(x, t)) = f(x)$  for all  $x \in L_0$  and all  $t$ , and  $\text{diam}[F(\{x\} \times [0, 1])] < 2\epsilon$  for all  $t$ . Now there exists an  $\epsilon$ -homotopy which pushes  $F(C \times \{1\})$  into  $X$  keeping  $F(B \times \{1\})$  fixed because of the fact that  $N_1$  is a strong  $(\epsilon, r)$ -deformation neighborhood. The homotopy  $g_t$  is the homotopy obtained by doing the two homotopies mentioned above in succession.  $\square$

**3. Inverse limits and CE limit maps.** In this section we give conditions under which a sequence of CE maps into neighborhoods of  $X$  converges to a CE map onto  $X$ . We first show that if  $X$  is defined as the intersection of a sequence of neighborhoods with sufficiently small collapses onto their spines, then there is a natural CE map from the inverse limit of the spines onto  $X$ . Then we construct a CE map from an inverse limit onto  $X$  as the limit of a sequence of level maps into that inverse sequence.

The following notation is assumed throughout this section:  $X$  is a compactum in the interior of the PL  $n$ -manifold  $M$ ,  $\{N_i\}$  is a sequence of neighborhoods of  $X$ ,  $N_{i+1} \subset \text{Int } N_i$  for each  $i$ ,  $X = \bigcap N_i$ ,  $N_i$  collapses to a compact polyhedron  $X_i$  via an  $\epsilon_i$ -collapse  $\xi_i$ , and  $\{\epsilon_i\}$  is a sequence of positive numbers which converges to 0. Let  $\rho_i: N_i \rightarrow X_i$  denote the  $\epsilon_i$ -retraction induced by  $\xi_i$  and let  $f_i: X_{i+1} \rightarrow X_i$  be the restriction of  $\rho_i$ . We will refer to the sequence  $\{N_i, X_i\}$  as a *defining sequence* for  $X$ . Notice that the defining sequence  $\{N_i, X_i\}$  implicitly defines the maps  $f_i$  and the numbers  $\epsilon_i$ .

**LEMMA 3.1.** *Suppose  $X$  is a compact subset of the PL  $n$ -manifold  $M$ . If  $\{N_i, X_i\}$  is a defining sequence for  $X$  such that  $\sum \epsilon_i < \infty$ , then there is a natural CE map  $G: \varprojlim \{X_i, f_i\} \rightarrow X$ .*

**REMARK.** It is possible to make  $G$  a homeomorphism by putting further restrictions on the defining sequence. To do so, the  $\epsilon_i$  would have to be chosen inductively and would depend on the preceding retractions. A proof of this stronger lemma would follow the lines of the proofs of Theorems 1 and 2 in [Br].

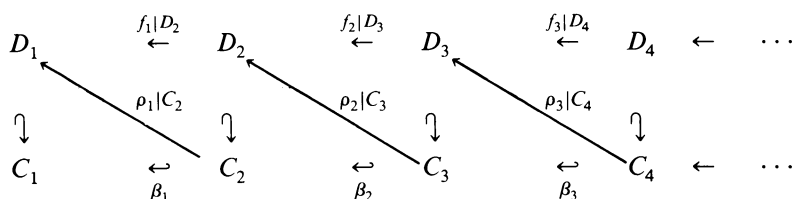
**PROOF OF LEMMA 3.1.** Let  $(x_i) \in \varprojlim \{X_i\}$ . The facts that  $d(x_i, x_{i+1}) < \epsilon_i$  and  $\sum \epsilon_i < \infty$  imply that the sequence  $\{x_i\}$  forms a Cauchy sequence when considered as a sequence of points in  $M$ . Therefore there is a unique point  $y$  such that  $x_i \rightarrow y$  as  $i \rightarrow \infty$ . Notice that  $y$  must be a point in  $X$ , so we can define  $G((x_i)) = y$ . It is clear that  $G$  is continuous, so the proof will be complete if we show that  $G^{-1}(x)$  is a nonempty cell-like set for each  $x \in X$ .

Fix  $x \in X$ . Define  $C_i = \{y \in N_i | d(y, x) < \sum_{k \geq i} \epsilon_k\}$  and  $D_i = C_i \cap X_i$ . It is clear that  $\bigcap C_i = \{x\}$ ; i.e., if  $\beta_i: C_{i+1} \rightarrow C_i$  denotes the inclusion map, then  $\{x\} = \varprojlim \{C_i, \beta_i\}$ . Pick  $y \in D_{i+1}$ ; then  $y \in X_{i+1}$  and  $d(x, y) < \sum_{k \geq i+1} \epsilon_k$ , so  $d(x, f_i(y)) < \sum_{k \geq i} \epsilon_k$ . Thus  $f_i(D_{i+1}) \subset D_i$ . We finish the proof by showing the following:  $G^{-1}(x) = \varprojlim \{D_i, f_i|D_{i+1}\}$  and  $\text{Sh}(\varprojlim \{D_i, f_i|D_{i+1}\}) = \text{Sh}(\varprojlim \{C_i, \beta_i\})$ . This will



complete the proof because then  $G^{-1}(x)$  is nonempty (it is the inverse limit of an inverse sequence of nonempty compacta) and it is cell-like because it has the shape of a point.

Consider the diagram:



There is an  $\varepsilon_i$ -homotopy from  $\beta_i$  to  $\rho_i|C_{i+1}$ . That homotopy stays in the  $\varepsilon_i$ -neighborhood of  $C_{i+1}$ , and thus stays in  $C_i$ . Therefore each of the lower triangles in the diagram commutes up to homotopy while each of the upper triangles commutes exactly. Thus we have a shape equivalence from  $\varprojlim \{D_i, f_i|D_{i+1}\}$  to  $\varprojlim \{C_i, \beta_i\}$ .

We now show that  $G^{-1}(x) = \varprojlim \{D_i, f_i|D_{i+1}\}$ . Pick  $(x_i) \in G^{-1}(x)$ . Then  $d(x_i, x_{i+1}) < \varepsilon_i$  and  $x_i \rightarrow x$ . Thus  $d(x_i, x) \leq \sum_{k \geq i} \varepsilon_k$  and  $x_i \in D_i$ . On the other hand, if  $(x_i) \in \varprojlim \{D_i\}$ , then  $d(x_i, x) \leq \sum_{k \geq i} \varepsilon_k$ . So  $x_i \rightarrow x$  as  $i \rightarrow \infty$  and  $(x_i) \in G^{-1}(x)$ .  $\square$

REMARK. Since every PL manifold has a metric in which closed balls are subpolyhedra, we may assume that the  $D_i$  in the proof above are subpolyhedra.

The following proposition is similar to Proposition 15 of [Hs]. The main difference is that we do not assume that the bonding maps in the inverse sequences are onto, but use Lemma 3.1 instead.

PROPOSITION 3.2. *Let  $X$  be a compact subset of the PL  $n$ -manifold  $M$  and let  $\{N_i, X_i\}$  be a defining sequence for  $X$  such that  $\sum \varepsilon_i < \infty$ . Suppose  $\{Z_i, g_i\}$  is an inverse sequence of compact polyhedra and that for each  $i$  there is a PL CE map  $\lambda_i: Z_i \rightarrow X_i$ . If  $f_i \lambda_{i+1}$  is  $\alpha_i$ -homotopic to  $\lambda_i g_i$  for each  $i$  and  $\sum \alpha_i < \infty$ , then  $\Lambda: \varprojlim \{Z_i, g_i\} \rightarrow X$  defined by  $\Lambda((z_i)) = \lim_{i \rightarrow \infty} \lambda_i(z_i)$  is a CE map.*

PROOF. Let  $Z = \varprojlim \{Z_i, g_i\}$ . Define  $\Lambda_j: Z \rightarrow N_1$  by  $\Lambda_j((z_i)) = \lambda_j(z_j)$ . Then

$$\begin{aligned}
 d(\Lambda_{j+1}((z_i)), \Lambda_j((z_i))) &= d(\lambda_{j+1}(z_{j+1}), \lambda_j(z_j)) = d(\lambda_{j+1}(z_{j+1}), \lambda_j g_j(z_{j+1})) \\
 &\leq d(\lambda_{j+1}(z_{j+1}), f_j \lambda_{j+1}(z_{j+1})) + d(f_j \lambda_{j+1}(z_{j+1}), \lambda_j g_j(z_{j+1})) \\
 &\leq \varepsilon_j + \alpha_j.
 \end{aligned}$$

Thus  $\{\Lambda_j\}$  forms a Cauchy sequence of maps and therefore converges to a continuous map  $\Lambda: Z \rightarrow X$ .

Fix  $x \in X$ . To complete the proof we must show that  $\Lambda^{-1}(x)$  is a nonempty cell-like subset of  $Z$ . Define  $D_i = \{y \in X_i | d(x, y) \leq \sum_{k \geq i} (\varepsilon_k + \alpha_k)\}$  and  $E_i = \lambda_i^{-1}(D_i)$ . Notice that  $f_i(D_{i+1}) \subset D_i$ . It is also true that  $g_i(E_{i+1}) \subset E_i$ . To see this, choose  $w \in E_{i+1}$ . We must show that  $g_i(w) \in E_i = \lambda_i^{-1}(D_i)$  which is equivalent to

showing that  $\lambda_i g_i(w) \in D_i$ .

$$\begin{aligned} d(\lambda_i g_i(w), x) &\leq d(\lambda_i g_i(w), f_i \lambda_{i+1}(w)) + d(f_i \lambda_{i+1}(w), x) \\ &\leq \alpha_i + [\varepsilon_i + d(\lambda_{i+1}(w), x)] \\ &\leq \alpha_i + \varepsilon_i + \sum_{k \geq i+1} (\varepsilon_k + \alpha_k) = \sum_{k \geq i} (\alpha_k + \varepsilon_k). \end{aligned}$$

Thus  $\lambda_i g_i(w) \in E_i$ .

Let  $G$  be the map defined in the proof of Lemma 3.1. We make the following three claims.

- (i)  $\Lambda^{-1}(x) = \varprojlim \{E_i, g_i|E_{i+1}\}$ .
- (ii)  $\text{Sh}(\varprojlim \{E_i, g_i|E_{i+1}\}) = \text{Sh}(\varprojlim \{D_i, f_i|D_{i+1}\})$ .
- (iii)  $\varprojlim \{D_i, f_i|D_{i+1}\} = G^{-1}(x)$ .

Once claims (i)–(iii) are verified, the proof is complete.

Let  $w \in \Lambda^{-1}(x)$ . Then  $w = (w_i)$  and  $\lambda_i(w_i) \rightarrow x$  as  $i \rightarrow \infty$ . But

$$d(\lambda_{i+1}(w_{i+1}), \lambda_i(w_i)) \leq \alpha_i + \varepsilon_i$$

(by the calculation at the beginning of this proof) and so  $w_i \in E_i$ . On the other hand, if  $(w_i) \in \varprojlim \{E_i\}$ , then  $d(\lambda_i(w_i), x) \leq \varepsilon_i + \alpha_i$ , so  $\lambda_i(w_i) \rightarrow x$  and  $(w_i) \in \Lambda^{-1}(x)$ . Thus claim (i) is verified.

The following diagram is homotopy commutative for each  $i$ .

$$\begin{array}{ccc} D_i & \xleftarrow{f_i|D_{i+1}} & D_{i+1} \\ \uparrow \lambda_i|E_i & & \uparrow \lambda_{i+1}|E_{i+1} \\ E_i & \xleftarrow{g_i|E_{i+1}} & E_{i+1} \end{array}$$

Furthermore,  $\lambda_i|E_i = \lambda_i|\lambda_i^{-1}(D_i)$  is a homotopy equivalence [La, Theorem 1.2]. Therefore  $\{\lambda_i|E_i\}$  induces a shape equivalence from  $\varprojlim \{E_i, g_i|E_{i+1}\}$  to  $\varprojlim \{D_i, f_i|D_{i+1}\}$  and claim (ii) is verified.

The proof of claim (iii) is just like the proof of the corresponding claim in the proof of Lemma 3.1. This completes the proof of Proposition 3.2.  $\square$

**4. A technical lemma.** Theorem 1 will be proved by constructing an infinite sequence of CE maps and then taking limits as in Proposition 3.2. The lemma in this section is the inductive step in that infinite construction. It is an  $\varepsilon$ -controlled version of [St] and corresponds to Proposition 6 of [Hs].

**LEMMA 4.1.** *Let  $X$  be a  $k$ -dimensional,  $LC^r$  compactum tamely embedded in  $\mathbf{R}^m$ ,  $m \geq 2k + 3$ ,  $r \leq k - 2$ , and let  $M$  be a PL  $n$ -manifold,  $n \geq k + 3$  and  $n \geq 2k - r$ . If  $\varepsilon_1, \varepsilon_2, \alpha_1$  and  $\alpha_2$  are positive numbers and*

$$X_1 \xleftarrow{\lambda_1} Z_1 \xrightarrow{\mu_1} Y_1$$

*satisfies*

(4.1)  $X_1$  is a  $k$ -dimensional polyhedron which is the spine of a strong  $(\varepsilon_1, r)$ -deformation neighborhood  $N_1$  of  $X$  in  $\mathbf{R}^m$ ;

- (4.2)  $N_1$  retracts to  $X_1$  via an  $\varepsilon_1$ -retraction  $\rho_1: N_1 \rightarrow X_1$ ;
  - (4.3)  $Z_1 = X_1 \cup M_1$ , where  $M_1$  is  $(r + 2)$ -dimensional;
  - (4.4)  $M_1 \cap X_1$  is  $(r + 1)$ -dimensional and collared in  $M_1$ ;
  - (4.5)  $\lambda_1: Z_1 \rightarrow X_1$  is a PL CE retraction with  $\lambda_1(M_1) \subset M_1 \cap X_1$ ;
  - (4.6)  $Y_1$  is a  $k$ -dimensional subpolyhedron of  $M$ ;
  - (4.7)  $\mu_1: Z_1 \rightarrow Y_1$  is a PL CE map;
  - (4.8) if  $A \subset X_1$  and  $\text{diam}[A] < \varepsilon_1$ , then  $\text{diam}[\mu_1(A)] < \alpha_1$ ;
  - (4.9) if  $A \subset N_1$  and  $\text{diam}[A] < \varepsilon_2$ , then  $\text{diam}[\mu_1\rho_1(A)] < \alpha_2$ ; and
  - (4.10)  $P_1$  is a regular neighborhood of  $Y_1$  and  $q: P_1 \rightarrow Y_1$  is a PL  $\alpha_1$ -retraction;
- then there exists

$$\begin{array}{ccccc}
 X_2 & \xleftarrow{\lambda_2} & Z_2 & \xrightarrow{\mu_2} & Y_2 \\
 \downarrow f & & \downarrow g & & \downarrow h \\
 X_1 & \xleftarrow{\lambda_1} & Z_1 & \xrightarrow{\mu_1} & Y_1
 \end{array}$$

which satisfies

- (4.11)  $X_2$  is a  $k$ -dimensional polyhedron which is the spine of a strong  $(\varepsilon_2, r)$ -deformation neighborhood  $N_2$  of  $X$  in  $N_1$ ;
- (4.12)  $f = \rho_1|_{X_2}$ ;
- (4.13)  $Z_2 = X_2 \cup M_2$ , where  $M_2$  is  $(r + 2)$ -dimensional;
- (4.14)  $M_2 \cap X_2$  is  $(r + 1)$ -dimensional and collared in  $M_2$ ;
- (4.15)  $\lambda_2: Z_2 \rightarrow X_2$  is a PL CE retraction with  $\lambda_2(M_2) \subset M_2 \cap X_2$ ;
- (4.16)  $Y_2$  is a  $k$ -dimensional polyhedron in  $P_1$ ;
- (4.17)  $h = q|_{Y_2}$ ;
- (4.18)  $\mu_2: Z_2 \rightarrow Y_2$  is a PL CE map;
- (4.19)  $\mu_2|_{X_2}$  is so close to  $\mu_1\rho_1|_{X_2}$  that if  $A \subset X_2$  and  $\text{diam}[A] < \varepsilon_2$ , then  $\text{diam}[\mu_2(A)] < \alpha_2$ ;
- (4.20)  $\lambda_1g$  is  $4(r + 1)\varepsilon_1$ -homotopic to  $f\lambda_2$  in  $K_1$ ; and
- (4.21)  $\mu_1g$  is  $8(r + 1)\alpha_1$ -homotopic to  $h\mu_2$  in  $Y_1$ .

PROOF. We may assume that  $\varepsilon_2 \leq \varepsilon_1$ . Let  $N_2$  be a strong  $(\varepsilon_2, r)$ -deformation neighborhood of  $X$  in  $N_1$  which  $\varepsilon_2$ -collapses to a  $k$ -dimensional polyhedron  $X_2$ . Let  $\rho_2: N_2 \rightarrow X_2$  be the  $\varepsilon_2$ -retraction induced by the collapse of  $N_2$  to  $X_2$  and let  $f = \rho_1|_{X_2}$ .

We next construct  $Z_2$ . It will be constructed in two steps. First we will lift the attaching map for  $M_1$  to  $X_2$  and form  $Z'$  by attaching  $M_1$  to  $X_2$ . Then  $Z_2$  will be formed by using an  $\varepsilon$ -controlled version of Stallings [St] to attach further mapping cylinders to  $Z'$ . The details follow.

Let  $\beta: M_1 \cap X_1 \rightarrow X_1$  denote the inclusion map. Since  $N_1$  is an  $(\varepsilon_1, r)$ -deformation neighborhood, there exists  $\beta': M_1 \cap X_1 \rightarrow X_2$  such that  $\beta'$  is  $2\varepsilon_1$ -homotopic to  $\beta$  in  $N_1$ . Define  $Z' = X_2 \cup_{\beta'} M_1$ . Notice that  $f\beta'$  is  $4\varepsilon_1$ -homotopic to  $\beta$  in  $X_1$ . Thus there is a map  $g: Z' \rightarrow Z_1$  such that  $g|_{X_2} = f$  and  $d(x, g(x)) < 4\varepsilon_1$  for every  $x \in M_1$ . The map  $g$  is just  $f$  on  $X_2$ , the identity on  $M_1$  minus a small collar on

$M_1 \cap X_1$  and stretches the collar out over the homotopy from  $\beta$  to  $f\beta'$ . Furthermore,  $\lambda_1$  induces a retraction  $\lambda': Z' \rightarrow X_2$  such that  $\lambda_1 g$  and  $f\lambda'$  are  $4\epsilon_1$ -homotopic in  $X_1$ .

Consider  $\mu_1 g: Z' \rightarrow Y_1 \subset P_1$ . Let  $g': Z' \rightarrow P_1$  be a PL general position map which is  $\alpha_1$ -homotopic to  $\mu_1 g$ . Let  $\Sigma$  denote the singular set of  $g'$  and let  $M'$  denote the mapping cylinder of  $g'|_\Sigma: \Sigma \rightarrow g'(\Sigma)$ . Then  $\dim \Sigma \leq r$  and  $\dim M' \leq r + 1$ . We identify  $\Sigma$  with  $\Sigma \times \{0\} \subset M'$  and let  $\xi: M' \rightarrow g'(\Sigma)$  denote the map which collapses out the fibers of the mapping cylinder.

We now wish to extend the inclusion map  $\Sigma \subset Z'$  to a map of  $M'$  into  $Z'$ . The  $\alpha_1$ -homotopy from  $g'$  to  $\mu_1 g$  gives us a map  $\tau_1: M' \rightarrow P_1$  such that  $\tau_1|_\Sigma = \mu_1 g|_\Sigma$ ,  $\tau_1|_{g'(\Sigma)} = \text{inclusion}$  and the diameter of the image under  $\tau_1$  of each fiber of  $M'$  is less than  $\alpha_1$ . Then  $q\tau_1: M' \rightarrow Y_1$  and  $q\tau_1|_\Sigma = \mu_1 g|_\Sigma$ . Notice that the  $\alpha_1$ -homotopy from  $\tau_1$  to  $q\tau_1$  can be used to construct an  $\alpha_1$ -homotopy from  $q\tau_1$  to  $\xi$ . The homotopy is obviously not rel  $\Sigma$ , but it does have the property that, when restricted to  $\Sigma$ , it is just the reverse of the  $\alpha_1$ -homotopy from  $g'$  to  $\mu_1 g$ . Since  $\mu_1$  is a CE map, there exists a lift  $\tau_2: M' \rightarrow Z_1$  such that  $\tau_2|_\Sigma = g|_\Sigma$  and such that  $\mu_1 \tau_2$  is  $\alpha_1$ -homotopic to  $q\tau_1$  rel  $\Sigma$ . We may assume that  $\tau_2$  is PL. The next step is to lift to  $Z'$ . We define  $\tau_3: M' \rightarrow Z'$  as follows. First  $\tau_3|_\Sigma = \text{inclusion}$  and  $\tau_3|_{\tau_2^{-1}(M_1)} = g^{-1}\tau_2|_{\tau_2^{-1}(M_1)}$ . Then use Proposition 2.4 to extend  $\tau_3$  to all of  $M'$  in such a way that  $g\tau_3$  is  $5\epsilon_1$ -homotopic to  $\tau_2$  rel  $\Sigma$ .

Now define  $M'' = M' \times [0, 1]/\{x\} \times [0, 1] | x \in \Sigma\}$ ; i.e.,  $M''$  is obtained from  $M' \times [0, 1]$  by identifying each of the sets  $\{x\} \times [0, 1]$ ,  $x \in \Sigma$ , to a point. Notice that  $M''$  is  $(r + 2)$ -dimensional and contains two copies of  $M'$ :  $M' \times \{0\}$  and  $M' \times \{1\}$ . Roughly speaking, we will attach  $M''$  to  $Z'$  along one copy of  $M'$  to form  $Z_2$  and shrink out fibers in the other copy to form  $Y_2$ .

The map  $\tau_3$  defines, in a natural way, a map  $\tau: M' \times \{0\} \rightarrow Z'$ . Define  $Z'' = Z' \cup_\tau M'$ , define  $\xi': Z'' \rightarrow Z'$  to be the map which collapses out the interval factors in  $M''$  and define  $g'': Z'' \rightarrow Z_1$  by  $g'' = g\xi'$ . Then define  $Y'$  to be the polyhedron obtained from  $M''$  by shrinking out fibers of the mapping cylinder  $M' \times \{1\} \subset M''$ . Let  $\mu: Z'' \rightarrow Y'$  be the natural collapsing map.

We now claim that there is a map  $\pi: Y' \rightarrow P_1$  such that  $q\pi\mu$  is  $8\alpha_1$ -homotopic to  $\mu_1 g''$  in  $Y_1$ . Since the point preimages of  $g'$  are precisely the subsets of  $Z'$  which are identified by  $\mu$ , we can use  $g'$  to define  $\pi|_{Z'}$ . We can further extend  $\pi$  to  $Z' \cup M' \times \{1\}$  by making  $\pi(x, 1) = \xi(x)$  for each  $x \in M'$ . In order to extend  $\pi$  to the rest of  $M''$  we must find an  $8\alpha_1$ -homotopy of  $g'\tau_3$  to  $\xi$  which is rel  $\Sigma$ .

First, there is an  $\alpha_1$ -homotopy of  $g'\tau_3$  to  $\mu_1 g\tau_3$  (not rel  $\Sigma$ ). Next, there is a  $5\alpha_1$ -homotopy (this one rel  $\Sigma$ ) of  $\mu_1 g\tau_3$  to  $\mu_1 \tau_2$ . But then there is an  $\alpha_1$ -homotopy (rel  $\Sigma$ ) to  $q\tau_1$ . Finally,  $q\tau_1$  is  $\alpha_1$ -homotopic to  $\xi$ . All of this gives us an  $8\alpha_1$ -homotopy from  $g'\tau_3$  to  $\xi$ . On  $\Sigma$  this homotopy is not constant, but is the homotopy from  $g'|_\Sigma$  to  $\mu_1 g|_\Sigma$  followed by the reverse of that same homotopy. We can use a small collar on  $\Sigma$  in  $M'$  to taper off that homotopy and arrive at the homotopy we seek which is rel  $\Sigma$ . Thus  $\pi: Y' \rightarrow P_1$  is defined. We may assume that  $\pi$  is PL and in general position.

Notice that  $q\pi\mu: Z'' \rightarrow Y_1$  and  $\mu_1 g'': Z'' \rightarrow Y_1$  are  $8\alpha_1$ -homotopic. We can also define a CE map  $\lambda: Z'' \rightarrow X_2$  by  $\lambda = \lambda'\xi'$ . Then the two maps  $\lambda_1 g'': Z'' \rightarrow X_1$  and  $f\lambda: Z'' \rightarrow X_1$  are  $4\epsilon_1$ -homotopic in  $X_1$ .

*Special case:*  $k < (2/3)(n - 1)$ . In this case the dimension of the singular set of  $\pi$  is negative and thus  $\pi$  is an embedding. We take  $Z_2 = Z''$ ,  $g = g''$ ,  $Y_2 = \pi(Y')$ ,  $\mu_2 = \pi\mu$ ,  $\lambda_2 = \lambda$ , and  $h = q|Y_2$ . Since  $\mu_2|X_2 = g'|X_2$ , we can easily make sure that conclusion (4.19) is satisfied and the proof is complete in this case.

*General case.* In general, the dimension of the singular set of  $\pi$  is less than or equal to  $r - 1$ . We inductively apply the same construction to  $\mu\pi$  as was applied above to  $g'$ . This gives new polyhedra  $Z'''$  and  $Y''$  and a map  $\pi': Y'' \rightarrow P_1$ . This time we will only have to attach polyhedra of dimension  $(r - 1) + 2 = r + 1$ , and so the dimension of the singular set of  $\pi'$  will be only  $r - 2$ . This is repeated inductively until  $\pi$  is an embedding, which will take at most  $r$  repetitions.  $\square$

**5. Proof of Theorem 1.** In this section we use the lemmas from the preceding three sections to prove Theorem 1. The construction is based on that in the proof of Theorem 19 in [Hs].

Let  $X$ ,  $k$ ,  $r$ ,  $f$ ,  $M$ , and  $n$  be as in the statement of Theorem 1. We begin by embedding  $X$  as a tame subset of  $\mathbb{R}^{2k+3}$ . The map  $f: X \rightarrow M$  can be extended to a compact neighborhood  $U$  of  $X$  in  $\mathbb{R}^{2k+3}$ . We may assume that  $f: U \rightarrow M$  is PL and in general position. Fix a sequence  $\{\alpha_i\}$  of positive numbers such that  $\sum \alpha_i < \infty$ . Next choose a number  $\epsilon_1 > 0$  such that if  $A$  is a compact subset of  $U$  and  $\text{diam}[A] < \epsilon_1$ , then  $\text{diam}[f(A)] < \alpha_1$ . Now let  $N_1$  be a strong  $(\epsilon_1, r)$ -deformation neighborhood of  $X$  in  $U$ . Let us say that  $N_1$   $\epsilon_1$ -collapses to the  $k$ -dimensional polyhedron  $X_1$  and that the collapse induces the  $\epsilon_1$ -retraction  $\rho_1: N_1 \rightarrow X_1$ .

Since  $N_1$  is a strong  $(\epsilon_1, r)$ -deformation neighborhood of  $X$ ,  $f|X_1: X_1 \rightarrow M$  is  $(r + 1)$ -connected. We can therefore apply Stallings' construction [St] to  $f|X_1$ . This gives us two  $k$ -dimensional polyhedra  $Z_1 \supset X_1$  and  $Y_1 \subset M$ , a PL CE retraction  $\lambda_1: Z_1 \rightarrow X_1$  and a PL CE map  $\mu_1: Z_1 \rightarrow Y_1$ . Furthermore,  $Z_1$  consists of  $X_1$  with some mapping cylinders attached. In particular, we can write  $Z_1 = X_1 \cup M_1$ , where  $M_1$  is  $(r + 2)$ -dimensional,  $M_1 \cap X_1$  is  $(r + 1)$ -dimensional, and  $M_1 \cap X_1$  is collared in  $M_1$ . We can also arrange that  $\mu_1|X_1$  is arbitrarily close to  $f|X_1$ , and so we choose  $\mu_1|X_1$  to be so close to  $f|X_1$  that if  $A \subset X_1$  and  $\text{diam}[A] < \epsilon_1$ , then  $\text{diam}[\mu_1(A)] < \alpha_1$ .

We will next apply Lemma 4.1, but we must first specify the number  $\epsilon_2$  of the lemma. Choose  $\epsilon_2 > 0$  such that if  $A \subset N_1$  and  $\text{diam}[A] < \epsilon_2$ , then  $\text{diam}[\mu_1\rho_1(A)] < \alpha_2$ . Now apply Lemma 4.1. This gives us a new diagram

$$\begin{array}{ccccc} X_2 & \xleftarrow{\lambda_2} & Z_2 & \xrightarrow{\mu_2} & Y_2 \\ \downarrow f_1 & & \downarrow g_1 & & \downarrow h_1 \\ X_1 & \xleftarrow{\lambda_1} & Z_1 & \xrightarrow{\mu_1} & Y_1 \end{array}$$

as well as an  $\epsilon_2$ -neighborhood  $N_2$  of  $X_2$ .

Let  $\varepsilon_3 > 0$  be a number such that if  $A \subset N_2$  and  $\text{diam}[A] < \varepsilon_3$ , then  $\text{diam}[\mu_2 \rho_2(A)] < \alpha_3$ . Apply Lemma 4.1 again, this time to  $X_2$ ,  $Z_2$ , and  $Y_2$ . This procedure is continued inductively and produces an infinite diagram:

$$\begin{array}{ccccc}
 \vdots & & \vdots & & \vdots \\
 \downarrow & & \downarrow & & \downarrow \\
 X_4 & \xleftarrow{\lambda_4} & Z_4 & \xrightarrow{\mu_4} & Y_4 \\
 \downarrow f_3 & & \downarrow g_3 & & \downarrow h_3 \\
 X_3 & \xleftarrow{\lambda_3} & Z_2 & \xrightarrow{\mu_3} & Y_3 \\
 \downarrow f_2 & & \downarrow g_2 & & \downarrow h_2 \\
 X_2 & \xleftarrow{\lambda_2} & Z_2 & \xrightarrow{\mu_2} & Y_2 \\
 \downarrow f_1 & & \downarrow g_1 & & \downarrow h_1 \\
 X_1 & \xleftarrow{\lambda_1} & Z_1 & \xrightarrow{\mu_1} & Y_1
 \end{array}$$

We can make sure that the  $\varepsilon_i$  are chosen in such a way that  $\sum \varepsilon_i < \infty$ . Let  $Z = \varprojlim \{Z_i, g_i\}$  and let  $Y = \bigcap P_i$ . By Proposition 3.2 there are CE maps  $\lambda: Z \rightarrow X$  and  $\mu: Z \rightarrow Y$ . This completes the construction of  $Y$ . To finish the proof of the theorem we need only show that  $Y$  is  $\text{LC}'$ . To do so, we prove the following statement:

*Suppose  $L$  is an  $(r+1)$ -dimensional polyhedron and  $L_0$  is a subpolyhedron of  $L$ . For every map  $F_0: L_0 \rightarrow Y_{i+1}$  such that  $h_i F_0$  extends to  $F: L \rightarrow Y_i$  there exists an extension  $F': L \rightarrow Y_{i+1}$  of  $F_0$  such that  $h_i F'$  is  $[5 + 12(r+1)]\alpha_i$ -homotopic to  $F \text{ rel } L_0$  in  $Y_i$ .*

This is really just the statement that Proposition 2.4 holds for the neighborhoods  $\{P_i\}$  of  $Y$ . It is easy to see that this implies that  $Y$  is  $\text{LC}'$ : just take a small singular  $m$ -sphere in  $Y$ ,  $m \leq r$ , homotope it to a point in a small subset of  $M$ , and then apply the statement above to push that homotopy into  $Y$  by pushing it into smaller and smaller neighborhoods of  $Y$  with a Cauchy sequence of maps.

We will indicate how to get the extension  $F'$  whose existence was claimed above. It is constructed by using the CE maps  $\lambda_i$  and  $\mu_i$  to make the transition from  $Y$  to  $X$  and back again. We leave the calculation of the size of the homotopies involved to the reader.

Since  $\mu_{i+1}$  is CE, we can lift  $F_0$  to a map  $F_1: L_0 \rightarrow Z_{i+1}$  such that  $\mu_{i+1} F_1$  is arbitrarily close to  $F_0$ . We choose it to be so close that if we can extend the map  $\mu_{i+1} F_1$ , then we are done. Consider  $\mu_i g_i F_1: L_0 \rightarrow Y_i$ . This map is  $8(r+1)\alpha_i$ -homotopic to  $h_i \mu_{i+1} F_1$  which extends to all of  $L$  (via  $F$ ) and thus there is an extension  $F_2: L \rightarrow Z_i$  of  $g_i F_1$ . Now consider  $\lambda_{i+1} F_1: L_0 \rightarrow X_{i+1}$ . Since  $f_i \lambda_{i+1} F_1$  is  $4\varepsilon_1$ -homotopic to  $\lambda_i g_i F_1$  and  $\lambda_i g_i F_1$  extends to  $\lambda_i F_2$ , we have that  $f_i \lambda_{i+1} F_1$  extends to  $F_3: L \rightarrow X_i$ . Thus we are in a position to apply Proposition 2.4 to the maps  $\lambda_{i+1} F_1: L_0 \rightarrow X_{i+1}$  and  $F_3: L \rightarrow X_i$  which extends  $f_i \lambda_{i+1} F_1$ . Thus there is an

extension  $F_4: L \rightarrow X_{i+1}$  of  $\lambda_{i+1}F_1$  to all of  $L$ . Now use the fact that  $\lambda_{i+1}$  is CE to lift  $F_4$  to a map  $F_5: L \rightarrow Z_{i+1}$  which extends  $F_1$ . Now  $\mu_{i+1}F_5$  is the desired extension of  $\mu_{i+1}F_1$ .  $\square$

REMARK 5.1. Notice that (in the proof above)  $\varprojlim \{X_i, f_i\} \subset \varprojlim \{Z_i, g_i\}$ . So if we did the extra work necessary to make the map  $G$  of Lemma 3.1 a homeomorphism, we would have  $X \subset Z$  and  $\lambda: Z \rightarrow X$  would be a retraction.

REMARK 5.2. It might seem more natural to assume in the statement of Theorem 1 that  $f$  is  $(r+1)$ -shape connected rather than  $(r+1)$ -connected. But the two assumptions are equivalent because either way we get that the extended map  $f$  has the property that  $f|X_i: X_i \rightarrow M$  is  $(r+1)$ -connected (cf. [Hs, Proposition 14]).

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