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HOMOTOPY EQUIVALENCES ON 3-MANIFOLDS

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ABSTRACT. Suppose M^3 is a 3-manifold and $f\colon M^3 \to X$ is a homotopy equivalence onto an ANR X. In this paper the cellularity properties of point preimages under f are studied. It is shown that for every open cover α of X there exists an open cover β of X such that if f is a β-equivalence then each $f^{-1}(x)$ is α-cellular in $M^3_+ \times \mathbf{R}^1$. In fact, the (open) cellularity occurs in a continuous fashion and so the map f can be approximated by a Euclidean bundle map.

0. Introduction. Suppose M^n is an n-manifold, X is an ANR and $f\colon M^n \to X$ is a map. The questions studied in this paper concern the relationship between homotopy properties of f and the structure of point inverses under f, particularly in dimension $n = 3$.

The case in which f is a cell-like map (or equivalently, f is a fine homotopy equivalence [**3**]) is well understood. Each point inverse is a cell-like set and McMillan [**5**] has given a criterion for determining whether or not these cell-like sets are actually cellular as long as $n \geq 5$. Freedman [**2**] has recently extended that result to the case $n = 4$. It follows that for every value of n, f is cell-like if and only if each $f^{-1}(x)$ is cellular in $M_+ \times \mathbf{R}^1$. (Here M_+ denotes M plus a collar attached along the boundary of M.) In addition, Rushing [**7**] has shown that if $n \geq 5$, then f can always be approximated by an $(n+1)$-disk bundle map.

In [**6**], Montejano and Rushing generalize the preceding results to the case in which f is an α-equivalence. Their theorems are true generalizations since a cell-like map of ANR's is an α-equivalence for every open cover α of the range while not every α-equivalence is a cell-like map. The proofs given by Montejano and Rushing are valid only in dimensions ≥ 4, and it is the purpose of this paper to prove the low-dimensional cases of their theorems. Since the case $n \leq 2$ is trivial, that means that we concentrate on the case $n = 3$.

Before stating the main theorem, we give a definition.

DEFINITION [**6**]. Let $E \subset M^n$ be a closed subset of an n-manifold M, let B be a metric space and let α be an open cover of B. A map $p\colon E \to B$ is said to be *α-cellular* (in M^n) if for every $b \in B$ and $\varepsilon > 0$, there exists an n-cell $D \subset M^n$ such that $D \cap E \subset p^{-1}(N_\alpha(b))$ and

$$p^{-1}(b) \subset \operatorname{Int} D \subset D \subset N_\varepsilon(p^{-1}(N_\alpha(b))) \subset M^n.$$

Notice that $f\colon M^n \to X$ is cellular if and only if it is α-cellular for every open cover α of X.

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The first theorem is the 3-dimensional case of [**6**, Theorem 2]. We repeat that M^3_+ denotes M^3 with a collar attached along its boundary.

THEOREM 1. *Let X be an ANR. For every open cover α of X there exists an open cover β of X such that if $f\colon M^3 \to X$ is a β-equivalence from a 3-manifold onto X, then f is α-cellular in $M^3_+ \times \mathbf{R}^1$.*

It should be pointed out explicitly that we are only proving the existence of one 4-cell neighborhood, not a whole sequence of them. Since every 3-manifold is triangulable, we will assume from now on that M^3 is a piecewise linear (PL) manifold.

REMARK. Even though $M^3_+ \times \mathbf{R}^1$ has a PL structure, we will only show that f is α-cellular with respect to topological 4-balls. The reason is that we will apply Freedman [**2**]. If the 3-dimensional Poincaré Conjecture were known to be true, this appeal to [**2**] would be unnecessary and we could conclude that f is α-cellular with respect to PL 4-balls.

In [**6** and **7**], microbundle techniques are used to patch the interiors of the cells of Theorem 1 together into a continuous collection and then a continuous collection of closed cells is obtained because a certain obstruction vanishes by work of Kirby and Siebenmann. In low dimensions we do not arrive at a continuous collection of closed cell neighborhoods, but only at a continuous collection of Euclidean neighborhoods, and so the next theorem is only a partial analogue of [**6**, Theorem 2′].

DEFINITION. Let $E \subset M^n$ be a closed subset of an n-manifold M, let B be a metric space and let α be an open cover of B. A map $p\colon E \to B$ is said to be *α-euclidean* (in M^n) if for every $b \in B$ and $\varepsilon > 0$ there exists an open n-cell $U_b \subset M$ such that

$$U_b \cap E \subset p^{-1}(N_\alpha(b)) \quad \text{and} \quad p^{-1}(b) \subset U_b \subset N_\varepsilon(p^{-1}(N_\alpha(b))) \subset M.$$

The map p is *continuously α-euclidean* (in M^n) if p is α-euclidean and for every $\delta > 0$ there exists an open cover η of B such that $d(b,c) < \eta$ implies there exists a δ-homeomorphism $h\colon U_b \to U_c$.

THEOREM 2. *Let N^n be a closed n-manifold, $n \leq 3$. For every open cover α of N there exists an open cover β of N such that if $f\colon M^3 \to N$ is a β-equivalence from a 3-manifold onto N, then f is continuously α-euclidean in $M^3_+ \times \mathbf{R}^1$.*

Our final theorem is a 3-dimensional restricted version of [**6**, Theorem 1].

THEOREM 3. *Let N^n, $n \leq 3$, be a closed n-manifold. For every open cover α of N there exists an open cover β of N such that any β-equivalence $f\colon M^3 \to N$ from a compact 3-manifold onto N extends to a 4-euclidean bundle map $f'\colon M' \to N$ where $d(fr, f') < \alpha$ for some retraction $r\colon M' \to M$.*

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1. Definitions. In this section we define some of the terms used in the paper. All the definitions are standard but are collected here for the reader's convenience.

We use the abbreviation ANR for absolute neighborhood retract. A compact subset C of an ANR is said to be *cell-like* if C is contractible in every neighborhood of itself. This property is a topological property of C. A map $f\colon X \to Y$ of ANR's is a *cell-like map* if f is a proper surjection and $f^{-1}(y)$ is a cell-like subset of X for each $y \in Y$. A compact subset A of an n-manifold M^n is said to be *cellular* if there exists a sequence $D_1, D_2, D_3, \ldots$ of n-cells in M^n such that $A = \bigcap_{i=1}^{\infty} D_i$ and $D_{i+1} \subset \operatorname{Int} D_i$ for every i. The relationship between "cell-like" and "cellular" is that every cellular set is cell-like but the converse is not true; whether or not a cell-like subset of a manifold is cellular depends on how the set is embedded in the manifold.

Let B be a metric space. An *open cover* of B is simply a collection of open subsets of B whose union is all of B. If α is an open cover of B and $A \subset B$, then

$$N_\alpha(A) = \bigcup\{U \in \alpha \mid A \cap U \neq \emptyset\}.$$

A homotopy $f\colon X \times [0,1] \to B$ is said to be an *α-homotopy* if for every $x \in X$ there exists $U \in \alpha$ such that $f(\{x\} \times [0,1]) \subset U$. Suppose E is a second metric space and $f\colon E \to B$ is a surjection. We say that f is an *α-equivalence* [**1**] provided there exists a map $g\colon B \to E$ such that fg is homotopic to id_B via an α-homotopy and gf is homotopic to id_E via an $f^{-1}(\alpha)$-homotopy.

2. Building neighborhoods. Let $f\colon M^3 \to X$ be an α-equivalence. In this section we study the structure of neighborhoods of point preimages under f. The first result (Proposition 1) is that each $f^{-1}(x)$ has a small neighborhood in M^3_+ which consists of a homotopy 3-cell with 1-handles attached. It is then shown (using Freedman [**2**]) that each such neighborhood is contained in a small 4-cell in $M^3_+ \times \mathbf{R}^1$.

The first lemma follows from the technique of proof of Theorem 1 in [**4**] and is a version of Lemma 1 in [**5**]. For the sake of completeness we give a brief sketch of the proof here.

LEMMA 1. *Let K be a finite polyhedron in the interior of the compact, orientable manifold M_1. If each loop in K is null-homotopic in M_1, then there exists a compact 3-manifold B such that $K \subset \operatorname{Int} B \subset B \subset \operatorname{Int} M_1$ and $B = C \cup H$, where each component of ∂C is a 2-sphere and each component of H is a 3-cell which is attached to C as a 1-handle.*

PROOF. By taking a regular neighborhood if necessary, we may assume that K is a compact 3-manifold. Define

$$c(K) = \sum_{n \geq 0} n^2 g(n),$$

where $g(n)$ is the number of surfaces of genus n in ∂K. The proof is by induction on $c(K)$. If $c(K) = 0$, then we are done.

Suppose $c(K) > 0$ and K is reducible. (To say that K is reducible means that there exists a loop on ∂K which shrinks in K but does not shrink on ∂K; see [**4**].) Then there exists a PL disk $D \subset K$ such that $D \cap \partial K = \partial D$ and ∂D does not bound a disk on ∂K. Cut K open along D to obtain K'. Then $c(K') < c(K)$, so $K' \subset B'$ where B' is a compact 3-manifold as in the conclusion of the lemma. Now K is obtained from K' by attaching a single 1-handle, so it is possible to find the manifold B required to complete the proof by attaching a finite number of 1-handles to B'. (See [**4**, p. 512] for details.)

Next suppose $c(K) > 0$ and K is not reducible. In that case, the closure of $M_1 - K$ would have to be reducible. So we can add a nontrivial 2-handle to K in M_1 and reduce $c(K)$ in that way. □

LEMMA 2. *Suppose X is an ANR. For every open cover α of X there exists an open cover β of X such that if p: $Y \to X$ is a β-equivalence and $x \in X$, then $p^{-1}(N_\beta(x))$ can be shrunk to a point in $p^{-1}(N_\alpha(x))$.*

PROOF. Choose β' to be an open cover which star refines α and β'' to be an open cover such that each element of β'' is contractible in an element of β'. Let β be an open cover which star refines β''.

Suppose p: $Y \to X$ is a β-equivalence and $x \in X$. Then there exists an open set $B \in \beta''$ such that $x \in N_\beta(x) \subset B \subset N_\alpha(x)$. Let g: $X \to Y$ be a β-homotopy inverse for p, let i: $p^{-1}(N_\beta(x)) \to p^{-1}(N_\alpha(x))$ be the inclusion map, and let f_t: $B \to N_\alpha(x)$ be a homotopy such that $f_0 = \mathrm{id}$ and $f_1(B) = \{x\}$. If we use "$\simeq$" to denote "is homotopic to", then

$$i \simeq gp \mid p^{-1}(N_\beta(x)) = gf_0p \mid p^{-1}(N_\beta(x)) \simeq gf_1p \mid p^{-1}(N_\beta(x)).$$

But $gf_1p(p^{-1}(N_\beta(x)))$ is a single point. To complete the proof we need to show that the two homotopies keep $p^{-1}(N_\beta(x))$ inside $p^{-1}(N_\alpha(x))$.

Pick $y \in p^{-1}(N_\beta(x))$. There exists $U_1 \in \beta$ such that both $p(y)$ and x lie in U_1. Since the first homotopy is a $p^{-1}(\beta)$-homotopy, there also exists $U_2 \in \beta$ such that the track of y under the first homotopy is contained in $p^{-1}(U_2)$. But β is a star refinement of α, so there exists an open set $U \in \alpha$ such that $U_1 \cup U_2 \subset U$. Thus the track of y under the first homotopy lies in $p^{-1}(U) \subset p^{-1}(N_\alpha(x))$.

Choose $V_1 \in \beta'$ such that $f_t(B) \subset V_1$ for every t. To show that the second homotopy also keeps things inside of $p^{-1}(N_\alpha(x))$, it is enough to check that $g(V_1) \subset p^{-1}(N_\alpha(x))$. Pick $z \in V_1$. Since pg is β-homotopic to the identity, there exists $V_2 \in \beta$ such that both z and $pg(z)$ lie in V_2. So $p^{-1}(z) \cup p^{-1}(pg(z)) \subset p^{-1}(V_2)$. Because $V_1 \in \beta'$ and β' star refines α, there must be an open set $V \in \alpha$ such that $V_2 \cup V_1 \subset V$. Then

$$g(z) \in p^{-1}(pg(z)) \subset p^{-1}(V) \subset p^{-1}(N_\alpha(x)). \qquad \square$$

PROPOSITION 1. *Suppose X is an ANR and α is an open cover of X. Then there exists an open cover β of X with the following property: If p: $M^3 \to X$ is a β-equivalence from a 3-manifold onto X, $\varepsilon > 0$, and $x \in X$ then there is a compact 3-manifold $B_0 \subset M^3_+$ such that*

$$p^{-1}(x) \subset \operatorname{Int} B_0 \subset B_0 \subset N_\varepsilon(p^{-1}(N_\alpha(x))),$$

and $B_0 = C_0 \cup H_0$, where C_0 is a homotopy 3-cell and each component of H_0 is a 3-cell attached to C_0 as a 1-handle.

PROOF. Let $\beta_0 = \alpha$. Apply Lemma 2 four times to obtain open covers $\beta_1, \beta_2, \beta_3$ and β_4 of X such that β_i satisfies the conclusion of Lemma 2 relative to $\alpha = \beta_{i-1}$. Take $\beta = \beta_4$.

Suppose p: $M^3 \to X$ is a β-equivalence and fix $x \in X$, $\varepsilon > 0$. By the choice of β, there exist compact PL 3-manifolds M_1, M_2, M_3, M_4 and M_5 in M^3_+ such that

$$p^{-1}(x) \subset M_5 \subset M_4 \subset M_3 \subset M_2 \subset M_1 \subset N_\varepsilon(p^{-1}(N_\alpha(x)))$$

and each M_i is inessential in $\operatorname{Int} M_{i-1}$. Because M_i is inessential in M_{i-1}, each of M_2, M_3, M_4, and M_5 must be orientable. By Lemma 1 there exists a compact 3-manifold $B = C \cup H$ such that $M_4 \subset B \subset \operatorname{Int} M_3$, each component of ∂C is a 2-sphere, and each component of H is a 1-handle. Since M_3 shrinks in M_2, we can connect up the components of ∂C with arcs in M_2 which miss H. The proof of [**6**, Lemma 4] shows that we can push those arcs off M_5 and hence off $p^{-1}(x)$. Put the new arcs in general position with respect to ∂C. Choose subarcs $A_1, A_2, \ldots, A_n$ such that each A_i joins two different components of ∂C, $A_i \cap \partial C = \partial A_i$, and $\partial C \cup (A_1 \cup A_2 \cup \cdots \cup A_n)$ is connected. Let C_1 be obtained from C by adding a small regular neighborhood of A_1 to C if $\operatorname{Int} A_1 \cap C = \emptyset$ or else by subtracting the interior of a small regular neighborhood of A_1 from C if $\operatorname{Int} A_1 \subset C$. Notice that each component of ∂C_1 is still a 2-sphere and that the number of components has been reduced by one. Now A_2 either joins two different components of ∂C_1 or it does not. Define $C_2 = C_1$ if both ends of A_2 lie on the same component of ∂C_1. Else C_2 is obtained from C_1 by adding or subtracting a neighborhood of A_2 as appropriate. Continue inductively to define $C_3, C_4, \ldots, C_n$.

Let $C_0 = C_n$. Then ∂C_0 is a 2-sphere, $p^{-1}(x) \subset B_0 = C_0 \cup H$ and B_0 is a subset of M_2. To complete the proof we need only observe that C_0 can be shrunk in M_1 and that shrink can be cut off on $\partial C_0 = S^2$, so $\pi_1(C_0) = 0$. By duality C_0 is homologically trivial and so the Hurewicz Theorem shows that $\pi_i(C_0) = 0$ for all i. □

PROOF OF THEOREM 1. Choose β to be the open cover given by Proposition 1. Suppose p: $M^3 \to X$ is a β-equivalence, $x \in X$ and $\varepsilon > 0$.

By the choice of β (and the proof of Proposition 1) there exist compact 3-manifolds B_0 and M_1 in M^3_+ such that

$$M_1 \cap M^3 \subset p^{-1}(N_\alpha(x)),$$

$$p^{-1}(x) \subset \operatorname{Int} B_0 \subset B_0 \subset M_1 \subset N_\varepsilon(p^{-1}(N_\alpha(x))),$$

$$B_0 \text{ is inessential in } M_i \quad \text{and} \quad B_0 = C_0 \cup H_0,$$

where C_0 is a homotopy 3-cell and each component of H_0 is a 1-handle attached to C_0.

Identify B_0 and M_1 with $B_0 \times \{0\}$ and $M_1 \times \{0\}$ in $M^3_+ \times \mathbf{R}^1$. We show first that C_0 is cellular in $M^3_+ \times \mathbf{R}^1$. By [**2**, Theorem 1.11] it is enough to show that C_0 satisfies McMillan's cellularity criterion. Let π: $M^3_+ \times \mathbf{R}^1 \to M^3_+ \times \{0\}$ denote the projection and let U be a neighborhood of C_0 in $M^3_+ \times \mathbf{R}^1$. Choose V to be a smaller neighborhood such that $V \cap (M^3_+ \times \{0\})$ is connected and $\pi(V)$ is inessential in $U \cap (M^3_+ \times \{0\})$. Given a loop λ in $V - C_0$, put λ in general position. Then $\lambda \cap (M^3_+ \times \{0\})$ consists of a finite number of points. Since C_0 does not separate $V \cap (M^3_+ \times \{0\})$, those points can be joined pairwise by arcs in $(V \cap M^3_+ \times \{0\}) - C_0$. So λ is homotopic in $V - C_0$ to a finite product of loops, each of which is in either $M^3_+ \times [0, \infty)$ or in $M^3_+ \times (-\infty, 0]$. Because $\pi(V)$ shrinks in $U \cap (M^3_+ \times \{0\})$, each of these loops shrinks missing C_0 in either $U \cap (M^3_+ \times [0, \infty))$ or $U \cap (M^3_+ \times (-\infty, 0])$.

So there exists a topological 4-ball B^4 such that $C_0 \subset \operatorname{Int} B^4 \subset B^4 \subset M_1 \times [-\varepsilon, \varepsilon]$. We complete the proof by pushing B^4 out to cover H_0 as well. The core of each 1-handle in H_0 is an arc attached by its endpoints to C_0. For each such arc there is a homotopy pulling it through M_1 into C_0. Put those homotopies in general position in $M^3_+ \times \mathbf{R}^1$ and pipe the singularities off the edges of the resulting disks. In this way we obtain a finite collection of disjoint PL embedded disks in $M_1 \times [-\varepsilon, \varepsilon]$, attached to C_0 in such a way that each of the disks cancels one of the 1-handles of B_0. Simply push B^4 across those disks to obtain the desired 4-ball containing $p^{-1}(x)$. $\square$

3. Extending to a bundle map. In this section we indicate how Theorems 2 and 3 follow from Theorem 1. The proof of Theorem 1 given in this paper is definitely a low-dimensional proof and is different from that given by Montejano and Rushing [**6**] for the corresponding result in high dimensions. But the proofs of the other theorems are independent of dimension and have essentially been given in [**6** and **7**] already. Thus we will give only an outline here.

Suppose N^n is a closed n-manifold, $n \geq 3$, α is an open cover of N, M^3 is a 3-manifold and f: $M^3 \to N^n$ is a map. The first step is to adapt the proof of Theorem 1 to find an open cover β of N such that if f is a β-equivalence and Δ^n is a small tame n-cell in N, then $f^{-1}(\Delta^n)$ is α-cellular in $M^3_+ \times \mathbf{R}^1$; cf. [**6**, Lemma 6″]. (Here and elsewhere in this section the low-dimensional proofs would be simpler than the high-dimensional ones because the manifolds involved are automatically triangulable.) These cell neighborhoods are then used to find a microbundle neighborhood of the graph of f in $(M^3_+ \times \mathbf{R}^1) \times N$. This is done in the Fundamental Lemma of [**6**]. The next step is to use the Kister construction to find a Euclidean-bundle neighborhood of the graph inside the microbundle as in [**7**]. This completes the proof of Theorem 3. In high dimensions it is possible to go one step further and apply Kirby-Siebenmann to find a disk-bundle inside the Euclidean-bundle. But we do not have that tool available to us in this dimension.

Theorem 2 follows easily from Theorem 3. The argument is given on the last page of [**6**].

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